— LECTURE 4 — (01/29/2018)

Last time we concluded with a definition of the quantum plane (better to think about as a noncomm. algebra of the corresponding q-plane):

$$\boxed{k_q[x,y] := k\langle x,y\rangle/(yx-qxy)}$$

Rmks: (1) For $q=1$, we recover $k[x,y]$

(2) For any q, this algebra is graded via $\deg(x)=\deg(y)=1$

(3) The set $\{x^k y^l\}_{k,l\geq 0}$ is a basis of $k_q[x,y]$.

(4) For any k-algebra A: $\mathrm{Hom}_{Alg}(k_q[x,y], A) \overset{1\text{-to-}1}{\longleftrightarrow} \{(X,Y)\in A\times A \mid YX=qXY\}$

• Algebra $M_q(2)$

Assume $q^2\neq -1$. Consider q-commuting variables x,y and 4 more variables a,b,c,d which commute with both x and y. Define x',x'',y',y'' via

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \text{ and } \begin{pmatrix}x''\\y''\end{pmatrix}=\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \left(\text{equiv. } (x''\ y'')=(x\ y)\begin{pmatrix}a&b\\c&d\end{pmatrix}\right).$$

Prop 1: Assuming $q^2\neq -1$, $yx=qxy$, and $\{a,b,c,d\}$ commute with $\{x,y\}$, TFAE:

(i) $y'x'=qx'y'$ and $y''x''=qx''y''$

(ii) $ba=qab$, $ca=qac$, $db=qbd$, $dc=qcd$, $bc=cb$, $ad-da=(q^{-1}-q)bc$

▸ (i) $\Rightarrow$ (ii)

$y'x'=qx'y' \Leftrightarrow (cx+dy)(ax+by)=q(ax+by)(cx+dy)$

$\Leftrightarrow x^2\cdot ca+y^2\cdot db+xy\cdot cb+yx\cdot da = x^2\cdot qac+y^2\cdot qbd+xy\cdot qad+yx\cdot qbc \quad (1)$

Since we are working in the algebra $k\langle x,y,a,b,c,d\rangle/(yx-qxy, ay-ya, ax-xa, \text{etc}\rangle$, the equality (1) is equivalent to

$ca=qac$; $db=qbd$; $cb+qda=qad+q^2bc \Leftrightarrow ad-da=q^{-1}cb-qbc$.

$y''x''=qx''y'' \Leftrightarrow (bx+dy)(ax+cy)=q(ax+cy)(bx+dy) \Leftrightarrow$ (1) with $b\leftrightarrow c$

$\Leftrightarrow ba=qab$, $dc=qcd$, $ad-da=q^{-1}bc-qcb$.

Note: $q^{-1}cb-qbc=ad-da=q^{-1}bc-qcb \Rightarrow (q+q^{-1})bc=(q+q^{-1})cb \overset{q^2\neq-1}{\Longrightarrow} bc=cb$.

So: (i) $\Rightarrow$ (ii)

The opposite implication is straightforward.

Def: The algebra $M_q(2)$ is the quotient of the free algebra $k\langle a,b,c,d\rangle$ by the six relations of Prop 1 (ii):

$$\boxed{M_q(2) := k\langle a,b,c,d\rangle \Big/ \begin{pmatrix} ba-qab,\ ca-qac,\ db-qbd,\ dc-qcd, \\ bc-cb,\ ad-da-(q^{-1}-q)bc \end{pmatrix}}$$

Rmk: (a) In this definition q is arbitrary. In particular, for $q=1$ we get $k[a,b,c,d]$.

(b) This algebra is graded via $\deg(a)=\deg(b)=\deg(c)=\deg(d)=1$.

(c) For any algebra R, we have a natural bijection

$$\mathrm{Hom}_{Alg}(M_q(2), R) \xleftrightarrow{1\text{-to-}1} \left\{(A,B,C,D)\in R^4 \;\middle|\; \begin{matrix} BA=qAB,\ CA=qAC,\ DB=qBD, \\ DC=qCD,\ BC=CB,\ AD-DA=(q^{-1}-q)BC \end{matrix}\right\}$$

Such a quadruple $(A,B,C,D)\in R^4$ is called an <u>R-point of $M_q(2)$</u>.

(d) Writing such an R-point as $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, we see that the condition on being an R-point (assuming $q^2\neq -1$) is equivalent to $\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} X \\ Y\end{pmatrix}$ and $\begin{pmatrix} A & C \\ B & D\end{pmatrix}\begin{pmatrix} X \\ Y\end{pmatrix}$ to be R'-points, where $R' := R\langle X,Y\rangle/(YX-qXY)$, due to Prop 1.

Def: Define $\det_q\in M_q(2)$ via $\boxed{\det_q := ad-q^{-1}bc = da-qbc}$

<u>Lemma 1</u>: $\det_q$ is central

▶ $a\det_q = ada-qabc = ada-q\cdot q^{-1}\cdot q^{-1}\cdot bca = (ad-q^{-1}bc)a = \det_q\cdot a$

$b\det_q = bad-q^{-1}b^2c = q\cdot q^{-1}adb - q^{-1}bcb = (ad-q^{-1}bc)b = \det_q\cdot b$

$c\det_q = cad-q^{-1}cbc = q\cdot q^{-1}adc - q^{-1}bc\cdot c = (ad-q^{-1}bc)c = \det_q\cdot c$

$d\det_q = dad-q^{-1}dbc = dad-q^{-1}\cdot q\cdot q\cdot bcd = (da-qbc)d = \det_q\cdot d$

Since a,b,c,d generate $M_q(2)$, the result follows ◻

Def: Given an R-point of $M_q(2)$, written as $m=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, the element $\mathrm{Det}_q(m) := AD-q^{-1}BC = DA-qBC\in R$ is called the <u>quantum determinant</u> of m.

<u>Prop 2</u>: Let R be an algebra, $m=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, $m'=\begin{pmatrix} A' & B' \\ C' & D'\end{pmatrix}$ be two R-points of $M_q(2)$, s.t. $\{A,B,C,D\}$ commute with $\{A',B',C',D'\}$.

(i) The element $m'' := m'm$ is also an R-point of $M_q(2)$

(ii) $\mathrm{Det}_q(m'm) = \mathrm{Det}_q(m')\,\mathrm{Det}_q(m)$

(iii) The matrix $\begin{pmatrix} D & -qB \\ -q^{-1}C & A\end{pmatrix}$ is an R-point of $M_{q^{-1}}(2)$ and an R^{op}-point of $M_q(2)$.

Proof of Prop 2

(i) One could prove this in a long straightforward way, but we will rather use the interpretation via $\mathrm{Rmk}(d)$ (p.2). Let us write:

$$m'' = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = m'm \;\Rightarrow\; (m'')^t = \begin{pmatrix} A'' & C'' \\ B'' & D'' \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} = m^t (m')^t$$

Define $R' := R\langle X,Y\rangle/(YX-qXY)$. Then m'' is an R-point iff $m''\cdot\binom{X}{Y}$ and $(m'')^t\binom{X}{Y}$ are R'-points of the quantum plane.

But: (1) m is an R-point of $M_q(2) \Rightarrow Y'X' = qX'Y'$, where $\binom{X'}{Y'} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}\binom{X}{Y}$

m' is an R-point of $M_q(2)$ & (A',B',C',D') commute with $(A,B,C,D,X,Y) \Rightarrow$

$\Rightarrow m'\binom{X'}{Y'} = m''\binom{X}{Y}$ is an R-point of quantum plane.

(2) Likewise: $(m')^t\binom{X}{Y}$ is an R-point of quantum plane $\Rightarrow$

$\Rightarrow m^t(m')^t\binom{X}{Y} = (m'')^t\binom{X}{Y}$ is also an R-point of q-plane.

(ii) This can be done in a straightforward way:

$$\mathrm{Det}_q(m'm) = (A'A+B'C)(C'B+D'D) - q^{-1}(A'B+B'D)(C'A+D'C) \qquad \text{recall } A,B,C,D \text{ commute with } A',B',C',D'$$

$$= A'C'AB + A'D'AD + B'C'CB + B'D'CD - q^{-1}A'C'BA - q^{-1}A'D'BC - q^{-1}B'C'DA - q^{-1}B'D'DC$$

$$= (A'D'AD - q^{-1}A'D'BC) + \underbrace{(A'C'AB - q^{-1}A'C'BA)}_{\text{ZERO as } BA=qAB} + \underbrace{(B'D'CD - q^{-1}B'D'DC)}_{\text{ZERO as } DC=qCD} + (B'C'CB - q^{-1}B'C'DA)$$

$$\mathrm{Det}_q(m')\,\mathrm{Det}_q(m) = (A'D' - q^{-1}B'C')(AD - q^{-1}BC) = A'D'AD - q^{-1}A'D'BC - q^{-1}B'C'AD + q^{-2}B'C'BC$$

The equality $\mathrm{Det}_q(m'm) = \mathrm{Det}_q(m')\,\mathrm{Det}_q(m)$ follows now from

$$CB - q^{-1}DA = q^{-2}BC - q^{-1}AD \overset{CB=BC}{\Longleftrightarrow} AD - DA = (q^{-1}-q)BC \quad \checkmark$$

(iii) Set $\tilde{A} := D,\ \tilde{B} := -qB,\ \tilde{C} := -q^{-1}C,\ \tilde{D} := A$.

Then it is straightforward to see that 6 relations on (A,B,C,D) to be an R-point of $M_q(2)$ are equivalent to corresponding 6 relations on $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ to be an R-point of $M_{q^{-1}}(2)$ or R^{op}-point of $M_q(2)$ ∎

Rmk: One of the problems in the next homework will give a more conceptual proof of (ii).

- **Bialgebra structure on $M_q(2)$**

It turns out that $M_q(2)$ can be endowed with a bialgebra structure, s.t. coproduct and counit are given by the same formulas as for $q=1$ case.

Prop 3: (a) There exist algebra morphisms $\Delta: M_q(2) \to M_q(2)\otimes M_q(2)$, $\varepsilon: M_q(2)\to k$

s.t. $\Delta(a)=a\otimes a+b\otimes c$, $\Delta(b)=a\otimes b+b\otimes d$, $\Delta(c)=c\otimes a+d\otimes c$, $\Delta(d)=c\otimes b+d\otimes d$

$\varepsilon(a)=1$, $\varepsilon(b)=0$, $\varepsilon(c)=0$, $\varepsilon(d)=1$.

It is convenient to rewrite this as $\Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\varepsilon\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$

(b) We have $\Delta(\det_q)=\det_q\otimes\det_q$, $\varepsilon(\det_q)=1$

(c) The algebra $M_q(2)$ equipped with Δ,ε is a bialgebra.

! Note: $M_q(2)$ is neither commutative nor cocommutative.

▶ (a) According to Prop 2(i), the point $(\Delta(a),\Delta(b),\Delta(c),\Delta(d))$ is an $M_q(2)\otimes M_q(2)$-point of $M_q(2)$. Hence, the assignment Δ on generators uniquely extends to an algebra homomorphism $M_q(2)\to M_q(2)\otimes M_q(2)$.

Moreover, $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ is a k-point of $M_q(2)$. Hence the assignment ε on the generators uniquely extends to an algebra homomorphism $M_q(2)\to k$.

(b) $\varepsilon(\det_q)=\varepsilon(a)\varepsilon(d)-q^{-1}\varepsilon(b)\varepsilon(c)=1$.

The equality $\Delta(\det_q)=\det_q\otimes\det_q$ immediately follows from Prop 2(ii).

(c) Coassociativity axiom is easy to check using matrix form:

$$((\Delta\otimes \mathrm{Id})\Delta)\begin{pmatrix} a & b \\ c & d\end{pmatrix}=(\Delta\otimes\mathrm{Id})\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d\end{pmatrix}=(\mathrm{Id}\otimes\Delta)\Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

Counit axiom also follows from:

$$((\mathrm{Id}\otimes\varepsilon)\Delta)\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}=((\varepsilon\otimes\mathrm{Id})\Delta)\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Thus, $(M_q(2),\Delta,\varepsilon)$ is a coalgebra. However, since Δ,ε are algebra morphisms, due to part (a), we actually get a bialgebra structure on $M_q(2)$. ■

However, $M_q(2)$ is not a Hopf algebra, since there is no antipod.

To fix this, following the $q=1$ case from last time, we introduce $GL_q(2)$, $SL_q(2)$.

• **Hopf algebras $GL_q(2)$ and $SL_q(2)$**

Def: $\boxed{\begin{array}{l} GL_q(2) := M_q(2)[t]/(t\det_q - 1) \\ SL_q(2) := M_q(2)/(\det_q - 1) = GL_q(2)/(t-1) \end{array}}$

In particular, an R-point of $GL_q(2)$ (resp. $SL_q(2)$) is an R-point m of $M_q(2)$, s.t. $\mathrm{Det}_q(m)$ is invertible in R (resp. equal to 1).

Thm 1: (a) Formulas for Δ, ε from Prop 3 together with $\Delta(t) = t \otimes t$, $\varepsilon(t) = 1$ define a bialgebra structure on both $GL_q(2)$ and $SL_q(2)$.
(b) Moreover, both $GL_q(2)$ and $SL_q(2)$ are Hopf algebras with antipode S given by
$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \det_q^{-1} \cdot \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

▶ (a) To prove the existence of such Δ, ε for $SL_q(2)$, we appeal to Prop 3 and the following equalities:
$$\varepsilon(\det_q - 1) = 0, \quad \Delta(\det_q - 1) = \det_q \otimes \det_q - 1 \otimes 1 = (\det_q - 1) \otimes \det_q + 1 \otimes (\det_q - 1).$$
To prove the existence of such Δ, ε for $GL_q(2)$, we appeal to Prop 3 (and the coalgebra structure on $k[t]$) and the following equalities:
$$\varepsilon(t\det_q - 1) = 1 \cdot 1 - 1 = 0, \quad \Delta(t\det_q - 1) = t\det_q \otimes t\det_q - 1 \otimes 1 = (t\det_q - 1) \otimes t\det_q + 1 \otimes (t\det_q - 1)$$
Finally, the coassociativity and counit axioms follow from those for $M_q(2)$ and $k[t]$.

(b) According to Prop 2 (iii), $\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$ is an $M_q(2)^{op}$-point of $M_q(2)$.
Hence, there is an algebra homomorphism $\tilde{S}: M_q(2) \to M_q(2)^{op}$, s.t. $\tilde{S}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$. Set $\tilde{S}(t) = t$ to extend $\tilde{S}: M_q(2)[t] \to M_q(2)[t]^{op}$.
This descends to $\tilde{S}: GL_q(2) \to GL_q(2)^{op}$, $SL_q(2) \to SL_q(2)^{op}$, due to:
$$\tilde{S}(t\det_q - 1) = (\tilde{S}(d)\tilde{S}(a) - q^{-1}\tilde{S}(c)\tilde{S}(b))t - 1 = (ad - q^{-1}bc)t - 1 = t\det_q - 1$$
$$\tilde{S}(\det_q - 1) = \det_q - 1.$$

Recall: $\det_q$ is central in $GL_q(2)$, $SL_q(2)$ and invertible in both of them, we can define $S: GL_q(2) \to GL_q(2)^{op}$, $SL_q(2) \to SL_q(2)^{op}$ via
$$S(t) = t^{-1}, \quad S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det_q^{-1} \cdot \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} = \det_q^{-1} \cdot \tilde{S}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
To check that S is an antipode, it suffices to check $\sum_{(x)} S(x')x'' = \varepsilon(x) = \sum_{(x)} x' S(x'')$ for $x = a, b, c, d, t$. This follows from the matrix form:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} = \det_q \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad t \cdot t^{-1} = 1 = t^{-1}t$$

Rmk: (i) Note that $S^2\begin{pmatrix} a & b \\ c & d\end{pmatrix} := \begin{pmatrix} S^2(a) & S^2(b) \\ S^2(c) & S^2(d)\end{pmatrix} = \begin{pmatrix} a & q^2 b \\ q^{-2}c & d\end{pmatrix}$

$$\Downarrow$$

$$S^{2n}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & q^{2n} b \\ q^{-2n}c & d\end{pmatrix} = \begin{pmatrix} q^n & 0 \\ 0 & q^{-n}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} q^{-n} & 0 \\ 0 & q^n\end{pmatrix} \quad \forall n \in \mathbb{N}.$$

(ii) If $q = e^{\frac{2\pi i}{n}}$, then $GL_q(2), SL_q(2)$ are Hopf algebras with the order of S^2 equal to n.

- Coaction on q-plane

Last time we saw that the natural action $M(2), SL(2), GL(2) \curvearrowright k^2$, gives rise to $M(2), SL(2), GL(2)$ comodule-algebra structures on $k[x,y]$ with a coaction map given in matrix & vector form $\Delta_A\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \otimes \begin{pmatrix} x \\ y\end{pmatrix}$.

Thm 2: There exist a unique $M_q(2), SL_q(2), GL_q(2)$ comodule-algebra structures on the quantum plane $A = k_q[x,y]$ s.t.

$$\Delta_A(x) = a\otimes x + b\otimes y, \quad \Delta_A(y) = c\otimes x + d\otimes y.$$

In the matrix form: $\Delta_A\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \otimes \begin{pmatrix} x \\ y\end{pmatrix}$.

▶ First, we check that above assignment defines an algebra homom $A \xrightarrow{\Delta_A} M_q(2)\otimes A$

$$\Delta_A(y)\Delta_A(x) = (c\otimes x + d\otimes y)(a\otimes x + b\otimes y) = ca\otimes x^2 + db\otimes y^2 + cb\otimes xy + da\otimes yx$$

$$= qac\otimes x^2 + qbd\otimes y^2 + (cb + qda)\otimes xy = qac\otimes x^2 + qbd\otimes y^2 + (q^2bc + qad)\otimes xy$$

$$= q(a\otimes x + b\otimes y)(c\otimes x + d\otimes y) = q\cdot \Delta_A(x)\Delta_A(y).$$

The verification of Δ_A being a comodule structure is completely analogous to the verification of coassociativity and counity of $M_q(2)$.

Thus, Δ_A defines an $M_q(2)$ comodule-algebra structure on $k_q[x,y]$.

As $M_q(2) \twoheadrightarrow SL_q(2)$ is an algebra homom, we also get an $SL_q(2)$ comodule-algebra structure on $k_q[x,y]$ ∎

Rmk: Decomposing $k_q[x,y] = \bigoplus_{n\geq 0} k_q[x,y]_n$ w.r.t. the natural grading, we see that each $k_q[x,y]_n$ is a subcomodule of $k_q[x,y]$

Rmk: In the above proof of $\Delta_A(y)\Delta_A(x) = q\Delta_A(x)\Delta_A(y)$, we could insted appeal to Prop 1.