To answer the question posed last time as well as for the future use, let us discuss:

- q-binomial coefficients

For $n\in\mathbb{Z}_{>0}$, set $\boxed{(n)_q := \frac{q^n-1}{q-1} = 1+q+q^2+\ldots+q^{n-1}}$

For $n\in\mathbb{Z}_{\geq 0}$, define the q-factorial of n via $(0)!_q := 1$ and

$$\boxed{(n)!_q := (1)_q\,(2)_q \cdots (n)_q = \frac{(q-1)(q^2-1)\cdot\ldots\cdot(q^n-1)}{(q-1)^n}}$$

Finally, for $0\le k\le n$, we define the Gauss polynomials:

$$\boxed{\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q\cdot(n-k)!_q}}$$

Lemma 1: For $0\le k\le n$, we have:

(a) $\binom{n}{k}_q = \binom{n}{n-k}_q$

(b) $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$ ← See homework #3 for geometric interpretation of $\binom{n}{k}_q$ and Lemma 1

(c) $\binom{n}{k}_q \in \mathbb{Z}[q]$ and $\binom{n}{k}_q\big|_{q=1} = \binom{n}{k}$.

► (a) Obvious

(b) $\binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q = \frac{(n-1)!_q}{(k-1)!_q(n-k)!_q} + q^k\cdot\frac{(n-1)!_q}{(k)!_q(n-1-k)!_q} = \frac{(n-1)!_q}{(k)!_q\,(n-k)!_q}\left((k)_q + q^k(n-k)_q\right) = \frac{(n)!_q}{(k)!_q(n-k)!_q}$

The second equality is analogous

(c) Immediately follows from (b). ∎

Prop 1: Let $yx = qxy$. Then $\forall n\in\mathbb{N}$, we have $\boxed{(x+y)^n = \sum_{0\le k\le n}\binom{n}{k}_q x^k y^{n-k}}$ $(*)$

► We prove by induction on n. The cases $n=0,1$ are obvious.

Assume we know $(*)$ for $n<N$ and want to deduce $(*)$ for $n=N$.

$$(x+y)^N = (x+y)\cdot(x+y)^{N-1} = (x+y)\sum_{0\le k\le N-1}\binom{N-1}{k}_q x^k y^{N-1-k} =$$

$$= \sum_{0\le k\le N-1}\binom{N-1}{k}_q x^{k+1}y^{N-1-k} + \sum_{0\le k\le N-1} q^k\binom{N-1}{k}_q x^k y^{N-k} \underset{\text{assuming } \binom{N-1}{N}_q=\binom{N-1}{-1}_q=0}{=}$$

$$= \sum_{0\le k\le N}\left(\binom{N-1}{k-1}_q + q^k\binom{N-1}{k}_q\right)x^k y^{N-k} \overset{\text{Lemma}}{=} \sum_{0\le k\le N}\binom{N}{k}_q x^k y^{N-k}.$$

This completes the step of induction ∎

Now we can answer the q-n from last time, asking to compute $\Delta_A(x^k y^\ell)$, $A=k_q[x,y]$.

$$\Delta_A(x^k y^\ell) = (a\otimes x + b\otimes y)^k(c\otimes x + d\otimes y)^\ell = \sum_{r=0}^{k}\sum_{s=0}^{\ell} q^{(k-r)s}\, a^r b^{k-r} c^s d^{\ell-s}\otimes x^{r+s}y^{k+\ell-r-s}\cdot\binom{k}{r}_{q^2}\binom{\ell}{s}_{q^2}$$

Note: q^2 is due to the fact $yx=qxy$ together with $ba=qab$, $dc=qcd$.

• q-exponential

Later on we will need a notion of q-exponent:

$$\boxed{e_q(z) := \sum_{n\geq 0} \frac{z^n}{(n)!_q}}$$ ← here we assume that q is not a root of 1.

Lemma 2: Let $yx = qxy$. Then $e_q(x+y) = e_q(x)\, e_q(y)$.

$$e_q(x)\,e_q(y) = \sum_{\substack{k\geq 0\\ l\geq 0}} \frac{x^k}{(k)!_q}\cdot\frac{y^l}{(l)!_q} = \sum_{n\geq 0} \frac{1}{(n)!_q}\Big(\sum_{0\leq k\leq n} \binom{n}{k}_q x^k y^{n-k}\Big) \overset{\text{Prop 1}}{=\!=} \sum_{n\geq 0} \frac{(x+y)^n}{(n)!_q} = \exp_q(x+y)$$

Let us consider two linear endomorphisms $Z, \tau_q \in \mathrm{End}(k[[z]]$ or $k[z])$ defined by

$$\boxed{(Zf)(z) = z\cdot f(z), \quad (\tau_q f)(z) = f(qz).}$$

Then $\tau_q Z = q\cdot Z\tau_q$, i.e. (Z, τ_q) is an $\mathrm{End}(k[[z]])$-point of the q-plane.
In particular, applying Prop 1 to the pair $(y = a\tau_q,\ x = -Z\tau_q)$, we get:

$$\boxed{((a-Z)\tau_q)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q (Z\tau_q)^k \cdot (a\tau_q)^{n-k}}$$ ← equality in $\mathrm{End}(k[[z]]$ or $k[z])$.

Applying this equality to $1 \in k[[z]]$, we get:

Lemma 3: $$\boxed{(a-z)(a-qz)\dots(a-q^{n-1}z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\frac{k(k-1)}{2}} a^{n-k} z^k}$$

Using this equality, we can now evaluate $e_q(z)^{-1}$.

Lemma 4: The inverse of $e_q(z)$ is given by

$$\boxed{e_q(z)^{-1} = \sum_{n\geq 0} (-1)^n q^{\frac{n(n-1)}{2}} \frac{z^n}{(n)!_q}}$$

We will come back to the q-calculus later on when it will be needed further.

- Hopf algebra structure on universal enveloping $\mathcal{U}(\mathfrak{g})$ (general case)

$\mathfrak{g}$ – Lie algebra $\rightsquigarrow \mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/(xy - yx - [x,y] \mid x,y \in \mathfrak{g})$

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, hook, "\imath"] \end{tikzcd}$$

Rmk: $\mathcal{U}(\mathfrak{g})$ is not graded, but only filtered. However, $\mathrm{gr.}\,\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$ by PBW Thm. For the future proofs, we will need the following universal property of $\mathcal{U}(\mathfrak{g})$.

Prop 2: For any associative algebra A and any linear map $f: \mathfrak{g} \to A$, such that $f(x)f(y) - f(y)f(x) = f([x,y])$ ($\forall x, y \in \mathfrak{g}$), there exists a unique algebra morphism $F: \mathcal{U}(\mathfrak{g}) \to A$, making the following diagram commut.

$$\begin{tikzcd} & \mathcal{U}(\mathfrak{g}) \arrow[dr, "F"] & \\ \mathfrak{g} \arrow[ur, hook, "\imath"] \arrow[rr, "f"'] & & A \end{tikzcd}$$

As an immediate consequence of this universal property, we have:

Lemma 5: (a) For any Lie algebra morphism $f: \mathfrak{g}_1 \to \mathfrak{g}_2$, there exists a unique algebra morphism $\mathcal{U}(f): \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_2)$ fitting into the commutative diagram

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "f"] \arrow[d, hook, "\imath"] & \mathfrak{g}_2 \arrow[d, hook, "\imath"] \\ \mathcal{U}(\mathfrak{g}_1) \arrow[r, "\mathcal{U}(f)"] & \mathcal{U}(\mathfrak{g}_2) \end{tikzcd}$$

(b) If $f_1: \mathfrak{g}_1 \to \mathfrak{g}_2$, $f_2: \mathfrak{g}_2 \to \mathfrak{g}_3$, then $\mathcal{U}(f_2 \circ f_1) = \mathcal{U}(f_2) \circ \mathcal{U}(f_1) : \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_3)$

(c) For two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ let us endow $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with a component-wise Lie algebra structure. Then $\mathcal{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \simeq \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$.

▶ This is left as a simple quick exercise ■

Now we are ready to define a Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$. Consider the Lie algebra morphisms:

(1) $\mathrm{diag}: \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ $\quad x \longmapsto (x, x)$

(2) $0 : \mathfrak{g} \to 0$ $\quad x \longmapsto 0$

(3) $\mathrm{inv}: \mathfrak{g} \to \mathfrak{g}^{op}$ $\quad x \longmapsto -x$

These Lie algebra morphisms induce the algebra morphisms due to Lemma 5:

$\Delta := \mathcal{U}(\mathrm{diag}): \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, $\varepsilon := \mathcal{U}(0): \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(0) = k$, and

$S := \mathcal{U}(\mathrm{inv}): \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}^{op}) \cong \mathcal{U}(\mathfrak{g})^{op}$ (canonical isom-m)

Thm 1: The enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a cocommutative Hopf algebra with the coproduct Δ, counit ε, and antipod S defined above. Moreover, if $x_1, \dots, x_n \in \mathfrak{g}$, then

$$\Delta(x_1 \dots x_n) = x_1 \dots x_n \otimes 1 + 1 \otimes x_1 \dots x_n + \sum_{k=1}^{n-1} \sum_{\sigma\text{-shuffle}} x_{\sigma(1)} \dots x_{\sigma(k)} \otimes x_{\sigma(k+1)} \dots x_{\sigma(n)}.$$

▶ Coassociativity is due to commutativity of

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathrm{diag}"'] & \mathfrak{g}\oplus\mathfrak{g} \arrow[d, "\mathrm{Id}\oplus\mathrm{diag}"] \\ \mathfrak{g}\oplus\mathfrak{g} \arrow[r, "\mathrm{diag}\oplus\mathrm{Id}"'] & \mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g} \end{tikzcd}$$

Counity is due to commutativity of

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, "\simeq"'] \arrow[d, "\mathrm{diag}"] \arrow[dr, "\simeq"] & \\ 0\oplus\mathfrak{g} & \mathfrak{g}\oplus\mathfrak{g} \arrow[l, "0\oplus\mathrm{Id}"] \arrow[r, "\mathrm{Id}\oplus 0"'] & \mathfrak{g}\oplus 0 \end{tikzcd}$$

Cocommutativity is due to commutativity of

$$\begin{tikzcd} & \mathfrak{g}\oplus\mathfrak{g} \arrow[dr, "\tau_{\mathfrak{g},\mathfrak{g}}"] & \\ \mathfrak{g} \arrow[ur, "\mathrm{diag}"] \arrow[rr, "\mathrm{diag}"'] & & \mathfrak{g}\oplus\mathfrak{g} \end{tikzcd}$$

The condition on S to be an antipode can not be deduced as above, since multiplication $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is not induced by a Lie algebra morphism $\mathfrak{g}\oplus\mathfrak{g} \to \mathfrak{g}$. Instead, we check directly on $\mathfrak{g}$ (which suffices as $\mathfrak{g}$ generate $\mathcal{U}(\mathfrak{g})$):

$$\sum_{(x)} S(x')x'' = (-x)\cdot 1 + 1\cdot x = 0 = \varepsilon(x) = \sum_{(x)} x' S(x'') \quad \forall x \in \mathfrak{g}.$$

The formula for the coproduct is tautological (see coproduct on $T(V)$). ■

Rmks: (a) If $\mathrm{char}(k) = 0$, then the symmetrization map $S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, $x_1 \dots x_n \mapsto \frac{1}{n!}\sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)}$ is a coalgebra isomorphism.

(b) Recalling that a $\mathfrak{g}$-module is the same as an $\mathcal{U}(\mathfrak{g})$-module, we recover the standard formulas for the dual and the direct sum of $\mathfrak{g}$-representations:

- If $\mathfrak{g} \curvearrowright V$, then $\mathfrak{g} \curvearrowright V^*$ via $(x.\alpha)(v) = \alpha(-x.v)$
- If $\mathfrak{g} \curvearrowright V_1, V_2$, then $\mathfrak{g} \curvearrowright V_1 \otimes V_2$ via $x(v_1 \otimes v_2) = x(v_1) \otimes v_2 + v_1 \otimes x(v_2)$.

This was the general story similar to the way we endowed $k[G]$ with a Hopf algebra structure for any algebraic group. However, our next goal is to introduce the q-analogue of $\mathcal{U}(\mathfrak{g})$ for the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. Before we do that, let us recall basic results about $\mathcal{U}(\mathfrak{sl}_2)$.

Let us first recall the standard results about $\mathfrak{sl}_2$.

- $\mathfrak{sl}_2$, classical story

Basis $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $\quad [e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$.

- As a consequence of PBW theorem $\{e^i f^j h^k\}_{i,j,k\geq 0}$ form a basis of $U(\mathfrak{sl}_2)$
- If $\mathrm{char}(k)=0$, then the centre of $U(\mathfrak{sl}_2)$ is generated by Casimir $C=ef+fe+\frac{h^2}{2}$.

Exercise 1: Determine the centre of $U(\mathfrak{sl}_2)$ if $\mathrm{char}(k)=p$.

- If $\mathrm{char}(k)=0$, then any finite-dimensional $U(\mathfrak{sl}_2)$-module is semisimple, while the collection of simple nonzero finite-dimensional reps is parametrized by $n\geq 0$. For each n, the corresponding repr-n V_n is $(n+1)$-dimensional, contains $v\neq 0$ s.t. $e(v)=0$ & $h(v)=n\cdot v$ (we call $v\in V_n$ the highest weight vector)
- The Casimir C acts on V_n via $\frac{n(n+2)}{2}\mathrm{Id}$.
- For any $n\geq m\geq 0$, we have $V_n\otimes V_m\simeq V_{n+m}\oplus V_{n+m-2}\oplus\ldots\oplus V_{n-m+2}\oplus V_{n-m}$
- One can realize V_n as $k[x,y]_n$ – n^{th} graded component of $k[x,y]$. Thus $k[x,y]=\bigoplus_{n\geq 0}k[x,y]_n$ encodes each f.d. irreducible repr-n once. Here e,f,h act via $x\frac{\partial}{\partial y}$, $y\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$, respectively.

Moreover, the last picture fits in a broader context. Similarly to the way we defined H-comodule-algebra structure, we have the dual notion:

Def: Let H be a bialgebra, A – an algebra. A is called an H-module-algebra if
(a) there is an action $H\curvearrowright A$
(b) the multiplication and unit $A\otimes A\to A$, $k\to A$ are H-module morphisms

Using Sweedler's notation, this means: $x(ab)=\sum_{(x)}(x'a)(x''b)$, $x(1)=\varepsilon(x)1$.
In particular, we have:

Lemma 6: Given a Lie algebra $\mathfrak{g}$ and an algebra A, endowing A with an $U(\mathfrak{g})$-module-algebra structure is equivalent to endowing A with $\mathfrak{g}$-action via derivations of A, i.e. $x.(ab)=(x.a)\cdot b+a\cdot(x.b)$.

In this context, we see that $k[x,y]$ is an $U(\mathfrak{sl}_2)$-module-algebra

- Algebra $U_q(\mathfrak{sl}_2)$

Fix $q \in k \setminus \{\pm 1\}$. Define $[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]! := [1]\cdot[2]\cdot\ldots\cdot[n]$, $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!}$

Rmk: This is more symmetric than $(n)_q$ due to $[-n] = -[n]$, but they are related via $[n] = q^{1-n}(n)_{q^2}$, $[n]! = q^{-n(n-1)/2}(n)!_{q^2}$, $\begin{bmatrix} n \\ k \end{bmatrix} = q^{-k(n-k)}\binom{n}{k}_{q^2}$.

Def: $U_q(\mathfrak{sl}_2)$ is an associative algebra generated by $\{E, F, K, K^{-1}\}$ subject to the following defining relations:

$$\boxed{K^{-1}K = KK^{-1} = 1,\ KE = q^2 EK,\ KF = q^{-2} FK,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}}$$

Rmk: Similarly to $U(\mathfrak{sl}_2)$, there is a Cartan automorphism ω of $U_q(\mathfrak{sl}_2)$

$$\omega:\ E \mapsto F,\ F \mapsto E,\ K^{\pm 1} \mapsto K^{\mp 1}.$$

Question: In which sense can we view $U_q(\mathfrak{sl}_2)$ as a deformation of $U(\mathfrak{sl}_2)$? The problem is that while the first three rel-s are defined for any $q \in k$, the last doesn't make sense.

To fix this, we give an alternative presentation, in which one can specialize $q \mapsto 1$.

Def: Define $\tilde{U}_q(\mathfrak{sl}_2)$ to be the associative algebra, generated by $\{E, F, K, K^{-1}, L\}$ subject to the following defining relations;

$$KK^{-1} = K^{-1}K = 1,\ KE = q^2 EK,\ KF = q^{-2}FK,\ EF - FE = L,\ (q - q^{-1})L = K - K^{-1}$$
$$LE - EL = q(EK + K^{-1}E),\ LF - FL = -q^{-1}(FK + K^{-1}F).$$

Lemma 7: For $q \neq \pm 1$, $U_q(\mathfrak{sl}_2) \xrightarrow{\sim} \tilde{U}_q(\mathfrak{sl}_2)$ via $E \mapsto E, F \mapsto F, K^{\pm 1} \mapsto K^{\pm 1}$

Define $\phi: \tilde{U}_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ via $E \mapsto E, F \mapsto F, K^{\pm 1} \mapsto K^{\pm 1}, L \mapsto [E, F]$. The only nontrivial verification we need to check is that ϕ is compatible with the last two defining rel-s of $\tilde{U}_q(\mathfrak{sl}_2)$.

$$[\phi(L), \phi(E)] = [[E,F], E] = \frac{1}{q - q^{-1}}[K - K^{-1}, E] = \frac{(q^2 - 1)EK + (q^2 - 1)K^{-1}E}{q - q^{-1}} = q\big(\phi(E)\phi(K) + \phi(K^{-1})\phi(E)\big)$$

The other one is analogous

∎

However, the benefit of working with $\tilde{U}_q(\mathfrak{sl}_2)$ is that $q \mapsto 1$ makes sense.

Prop 3: For $q = 1$, we get $\tilde{U}_1(\mathfrak{sl}_2) \simeq U(\mathfrak{sl}_2)[K]/(K^2 - 1)$ $\left(\Rightarrow U(\mathfrak{sl}_2) \simeq \tilde{U}_1(\mathfrak{sl}_2)/(K - 1)\right)$

Exercise 2: Prove this proposition

(the homom. $\tilde{U}_1(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)[K]/(K^2 - 1)$ is via $E \mapsto eK, F \mapsto f, K^{\pm 1} \mapsto K, L \mapsto hK$)

Rmk: The moral of this algebra $\tilde{U}_q(sl_2)$ is that it provides an "integral form". Note that while our definition was for numeric q, we can also make a similar definition over $k(q)$. But one can not specialize $q \mapsto 1$ in rational f-s as we may get ZERO in denominator. For that reason, it is very useful to have an algebra $\tilde{U}$ over $k[q, q^{-1}]$, s.t. $\tilde{U} \otimes_{k[q,q^{-1}]} k(q)$ is our $k(q)$-version of $U_q(sl_2)$.
We will come back to this point later on and the following el-s will play an important role:

$$\boxed{[K; a] := \frac{Kq^a - K^{-1}q^{-a}}{q - q^{-1}} \quad \forall a \in \mathbb{Z}}$$

Thm 2: The set $\{F^a K^n E^b \mid a, b \geq 0, n \in \mathbb{Z}\}$ forms a k-basis of $U_q(sl_2)$.
We will prove this next time. The proof is based on technical lemma:

Lemma 8: For any $n \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$, we have:

(a) $K^n E^m = q^{2mn} E^m K^n$, $K^n F^m = q^{-2mn} F^m K^n$

(b) $[E, F^m] = [m] \cdot F^{m-1} \cdot [K; 1-m]$

(c) $[F, E^m] = -[m] \cdot E^{m-1} \cdot [K; m-1]$.