Last time: Introduced $U_q(sl_2)$

Recall: (1) $[n] := \frac{q^n - q^{-n}}{q-q^{-1}} = q^{1-n}\cdot(n)_{q^2}$, $[n]! = [1]\cdot\ldots\cdot[n] = q^{-n(n-1)/2}(n)!_{q^2}$, $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = q^{-k(n-k)}\binom{n}{k}_{q^2}$

(2) $[K;a] := \frac{Kq^a - K^{-1}q^{-a}}{q-q^{-1}}$

(3) Cartan automorphism ω of $U_q(sl_2)$: $E \mapsto F, F \mapsto E, K^{\pm1} \mapsto K^{\mp1}$

Lemma 1: For any $n \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$, we have:

(a) $K^n E^m = q^{2nm} E^m K^n$, $K^n F^m = q^{-2mn} F^m K^n$

(b) $[E, F^m] = [m]\cdot F^{m-1}\cdot [K; 1-m]$

(c) $[F, E^m] = -[m]\cdot E^{m-1}\cdot [K; m-1]$

▶ (a) Obvious. ✓

(b) Prove by induction. Base $m=0,1$ – clear ($m=1$ – last defining rel-n of $U_q(sl_2)$)

The induction step proceeds as follows:

$$EF^m = EF^{m-1}F \overset{\text{Induction Ass.}}{=\!=\!=} (F^{m-1}E + [m-1]F^{m-2}[K; 2-m])F = F^mE + F^{m-1}\frac{K-K^{-1}}{q-q^{-1}} +$$

$$+ [m-1]F^{m-2}\cdot\frac{Kq^{2-m} - K^{-1}q^{m-2}}{q-q^{-1}}F = F^mE + F^{m-1}\left(\frac{K-K^{-1}}{q-q^{-1}} + [m-1]\cdot\frac{Kq^{-m} - K^{-1}q^m}{q-q^{-1}}\right)$$

$$= F^mE + F^{m-1}\cdot\frac{(q-q^{-1})(K-K^{-1}) + (q^{m-1} - q^{-m+1})(Kq^{-m} - K^{-1}q^m)}{(q-q^{-1})^2} = F^mE + F^{m-1}[m]\cdot[K; 1-m] \checkmark$$

(c) Analogous! ▫

Using this result, we can now complete the "PBW theorem" which we ended with last time:

Thm 1: The set $\{F^aK^nE^b \mid a,b \geq 0, n \in \mathbb{Z}\}$ forms a k-basis of $U_q(sl_2)$.

▶ • First, we prove that they span all $U_q(sl_2)$. Let $V := \mathrm{span}_k\langle F^aK^nE^b\rangle$.

Claim: V is stable under left multiplication by $U_q(sl_2)$

▷ It suffices to check that it is stable under left multiplication by gen-s:

(1) $F\cdot F^aK^nE^b = F^{a+1}K^nE^b \in V$

(2) $K^{\pm1}\cdot F^aK^nE^b = q^{\mp2a}F^aK^{n\pm1}E^b \in V$

(3) $E\cdot F^aK^nE^b \overset{\text{Lemma 1}}{=\!=\!=} (F^aE + F^{a-1}\cdot[a]\cdot[K; 1-a])K^nE^b =$

$$= q^{-2n}F^aK^nE^{b+1} + [a]\cdot F^{a-1}\cdot\left(\frac{Kq^{1-a} - K^{-1}q^{a-1}}{q^{1-a} - q^{a-1}}K^n\right)E^b \in V$$

▫

As V is $U_q(sl_2)$ stable and $1 = F^0K^0E^0 \in V$, we get $V = U_q(sl_2)$

• Second, we prove that these elements are linearly independent. We follow one of the classical proofs of PBW for $U(\mathfrak{g})$ by constructing an action of $U_q(sl_2)$ on $k[x, y, z^{\pm1}]$

(Second part of proof of Thm 1)

Consider a polynomial ring $A = k[x, y, z^{\pm 1}]$ (localized by powers of z).
Consider the following three endomorphisms of A:

$$\boxed{\begin{aligned}
&\tilde{F}(y^a z^n x^b) := y^{a+1} z^n x^b \\
&\tilde{K}(y^a z^n x^b) := q^{-2a}\, y^a z^{n+1} x^b \\
&\tilde{E}(y^a z^n x^b) := q^{-2n}\, y^a z^n x^{b+1} + [a] \cdot y^{a-1} \cdot \frac{z q^{1-a} - z^{-1} q^{a-1}}{q - q^{-1}}\, z^n x^b
\end{aligned}}$$

First, we note that $\tilde{K}$ is invertible with $\tilde{K}^{-1}(y^a z^n x^b) = q^{2a}\, y^a z^{n-1} x^b$.

<u>Claim</u>: The 4 endomorphisms $(\tilde{F}, \tilde{E}, \tilde{K}, \tilde{K}^{-1})$ satisfy the defining rel-s of $U_q(sl_2)$

(1) $\tilde{K}\tilde{K}^{-1} = \tilde{K}^{-1}\tilde{K} = Id_A$ – obvious

(2)
$$\left.\begin{aligned}
\tilde{K}\tilde{F} &: y^a z^n x^b \mapsto q^{-2(a+1)}\, y^{a+1} z^{n+1} x^b \\
\tilde{F}\tilde{K} &: y^a z^n x^b \mapsto q^{-2a}\, y^{a+1} z^{n+1} x^b
\end{aligned}\right\} \Rightarrow \tilde{K}\tilde{F} = q^{-2}\tilde{F}\tilde{K}$$

(3)
$$\left.\begin{aligned}
\tilde{K}\tilde{E} &: y^a z^n x^b \mapsto q^{-2n-2a} \cdot y^a z^{n+1} x^{b+1} + [a] \cdot q^{-2a+2} \cdot y^{a-1} \cdot \frac{z^2 q^{1-a} - q^{a-1}}{q - q^{-1}}\, z^n x^b \\
\tilde{E}\tilde{K} &: y^a z^n x^b \mapsto q^{-2n-2a-2} \cdot y^a z^{n+1} x^{b+1} + [a] \cdot y^{a-1} \cdot q^{-2a} \cdot \frac{z q^{1-a} - z^{-1} q^{a-1}}{q - q^{-1}}\, z^{n+1} x^b
\end{aligned}\right\} \Rightarrow \tilde{K}\tilde{E} = q^2 \tilde{E}\tilde{K}$$

(4) $[\tilde{E}, \tilde{F}] = \tilde{E}\tilde{F} - \tilde{F}\tilde{E}$

$$\left.\begin{aligned}
\tilde{E}\tilde{F} &: y^a z^n x^b \mapsto q^{-2n}\, y^{a+1} z^n x^{b+1} + [a+1]\, y^a \cdot \frac{z q^{-a} - z^{-1} q^a}{q - q^{-1}}\, z^n x^b \\
\tilde{F}\tilde{E} &: y^a z^n x^b \mapsto q^{-2n}\, y^{a+1} z^n x^{b+1} + [a]\, y^a \cdot \frac{z q^{1-a} - z^{-1} q^{a-1}}{q - q^{-1}}\, z^n x^b
\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow [\tilde{E}, \tilde{F}] : y^a z^n x^b \mapsto y^a \left([a+1] \cdot \frac{z q^{-a} - z^{-1} q^a}{q - q^{-1}} - [a] \cdot \frac{z q^{1-a} - z^{-1} q^{a-1}}{q - q^{-1}} \right) z^n x^b.$$

It remains to use

$$[a+1]\frac{z q^{-a} - z^{-1} q^a}{q - q^{-1}} - [a] \cdot \frac{z q^{1-a} - z^{-1} q^{a-1}}{q - q^{-1}} = \frac{(q^{a+1} - q^{-a-1})(z q^{-a} - z^{-1} q^a) - (q^a - q^{-a})(z q^{1-a} - z^{-1} q^{a-1})}{(q - q^{-1})^2} =$$

$$= \frac{z(-q^{-2a-1} + q^{-2a+1}) - z^{-1}(q^{2a+1} - q^{2a-1})}{(q - q^{-1})^2} = \frac{z q^{-2a} - z^{-1} q^{2a}}{q - q^{-1}}$$

The equality $[\tilde{E}, \tilde{F}] = \frac{\tilde{K} - \tilde{K}^{-1}}{q - q^{-1}}$ follows! □

Thus, we have an action $U_q(sl_2) \curvearrowright A$, $E \mapsto \tilde{E}$, $F \mapsto \tilde{F}$, $K^{\pm 1} \mapsto \tilde{K}^{\pm 1}$.
In particular, $F^a K^n E^b(1) = y^a z^n x^b$. Since the latter el-s $\{y^a z^n x^b \mid a, b \geq 0, n \in \mathbb{Z}\}$ are linearly independent in A, we also get a linear independence of $\{F^a K^n E^b\}$ ■

<u>Corollary</u>: Let U_q^+, U_q^0, U_q^- be the subalg-s of $U_q(sl_2)$, generated by $\{E\}, \{K^{\pm 1}\}, \{F\}$, resp.
<u>Then</u>: $U_q^+ = k[E]$, $U_q^- = k[F]$, $U_q^0 = k[K^{\pm 1}]$ and mult: $U_q^- \otimes U_q^0 \otimes U_q^+ \to U_q(sl_2)$ is isom.

Exercise 1: For $m,n\geq 0$, verify

$$\boxed{E^mF^n=\sum_{i=0}^{\min(m,n)}\begin{bmatrix}m\\i\end{bmatrix}\begin{bmatrix}n\\i\end{bmatrix}\cdot[i]!\,F^{n-i}\prod_{j=1}^{i}[K;i-(m+n)+j]\cdot E^{m-i}}$$

Prop 1: The algebra $U_q(\mathfrak{sl}_2)$ has no zero divisors.

▸ Due to Theorem 1, given any element $u\in U_q(\mathfrak{sl}_2)$ it can be uniquely written as a linear combination of terms $\{F^a\cdot h\cdot E^b \mid a,b\geq 0,\ h\in U_q^0\}$. We order such terms first by a and then by b, in other words, $F^a\cdot h\cdot E^b > F^{a'}h'E^{b'}$ if $a>a'$ or $a=a'$ & $b>b'$. This allows to speak about the leading term.

Let F^ahE^b be the leading term of u $(h\neq 0)$
$F^{a'}h'E^{b'}$ be the leading term of v $(h'\neq 0)$ $\quad(*)$

Let us also introduce algebra automorphisms $\{\gamma_i\}_{i\in\mathbb{Z}}$ of U_q^0 given by $\gamma_i(K)=q^iK$. In particular, $\boxed{hE=E\gamma_2(h)\ \&\ hF=F\gamma_{-2}(h)\ \forall h\in U_q^0}$ $(**)$

Claim: Assuming $(*)$, the leading term of $u\cdot v$ is $F^{a+a'}\gamma_{-2a'}(h)\gamma_{-2b}(h')E^{b+b'}$.

▸ Using the above Exercise 1, we see that (use h_i to shorten notations)

$$(F^ahE^b)(F^{a'}h'E^{b'})=\sum_{0\leq i\leq \min(b,a')}F^ahF^{a'-i}h_iE^{b-i}h'E^{b'}\overset{(**)}{=\!=}$$

$$=\sum_{i=0}^{\min(b,a')}F^{a+a'-i}\gamma_{-2(a'-i)}(h)h_i\gamma_{-2(b-i)}(h')E^{b+b'-i}$$

The leading term corresponds to $i=0$, in which case $h_0=1$, and we get $F^{a+a'}\gamma_{-2a'}(h)\gamma_{-2b}(h')E^{b+b'}$. Finally, we note that taking products of other terms of u and v, we get "smaller" terms for the same reason □

As $U_q^0=k[K^{\pm1}]$ does not have zero divisors and γ_i-automorphisms, we get $uv\neq 0$ if $u\neq 0$, $v\neq 0$. ■

Rmk: We note that the algebra $U_q(\mathfrak{sl}_2)$ is graded via $\deg(E)=1$, $\deg(F)=-1$, $\deg(K^{\pm1})=0$.
Moreover $Ku=q^{2i}uK$ if $\deg(u)=i$.

Corollary: If q is not a root of unity, then the above grading arises as eigenspaces w.r.t. $\mathrm{Ad}(K): u\mapsto KuK^{-1}$.

• Finite dimensional representations of $U_q(\mathfrak{sl}_2)$ $(q \neq \sqrt{1})$

Def: (a) Given a $U_q(\mathfrak{sl}_2)$-rep-n V, an element $v \in V \setminus \{0\}$ is called a *highest weight vector* of *weight* λ if $E(v)=0$ and $K(v)=\lambda\cdot v$.

(b) V is called a highest weight module of highest weight λ if it is generated by a highest weight vector v of weight λ.

(Most of the results below admit much shorter proofs for alg. closed k)

Lemma 2: Let V be a fin. dim. $U_q(\mathfrak{sl}_2)$-module. Then E, F act nilpotently on V, i.e. $\exists N \in \mathbb{N}$, s.t. $E^N(V)=0$, $F^N(V)=0$.

• If k was algebraically closed, we could argue as follows. If E is not nilpotent, then $\exists \lambda \neq 0$, $v \in V\setminus\{0\}$, s.t. $E(v)=\lambda\cdot v$. But then $E(K^n(v)) = q^{-2n}\cdot\lambda\cdot K^n(v)$. As $q \neq \sqrt{1}$, we would get $\{v, Kv, K^2v, \ldots\}$ – infinitely many vectors with distinct eigenvalues (note $K^n(v) \neq 0$ as K acts by an invertible operator), which is impossible as $\dim V < \infty$. This would give a contradiction $\Rightarrow$ E-nilpotent. Analogously F-nilpotent.

• *General k*

Let us give another argument which works for any field k. First, we have the direct sum decomposition

$$(+)\quad \boxed{V = \bigoplus V_{(f)},\quad V_{(f)} = \{v \in V \mid f(K)^n v = 0 \text{ for } n >> 0\},}$$

sum is over all irreducible polynomials $f \in k[x]$

If $f \in k[x]$ is irreducible, then $f_a(x) := f(q^a x) \in k[x]$ is also irreducible.
As $KE = q^2EK \Rightarrow g(K)E = Eg(q^2K)$ $\forall g \in k[x]$, so we get $g(K)E^r = E^r g(q^{2r}K) = E^r g_{2r}(K)$.
In particular, we get $E^r V_{(f)} \subseteq V_{(f_{-2r})}$ $\forall r \geq 0$.

As $\dim V < \infty$, the sum in $(+)$ is finite. Hence, it suffices to prove that if $V_{(f)} \neq 0$, then $V_{(f_{-2r})} = 0$ for $r >> 0$ (since it would imply $E^r V_{(f)} = 0$ by above). Assume the contrary, i.e. there are infinitely many nonzero subspaces among $\{V_{(f_{-2r})}\}$. Since the sum in $(+)$ is direct, $V_{(f)} = V_{(g)} \Leftrightarrow g = \text{const}\cdot f$, $\dim V < \infty$, we see that $\exists\ 0 \leq r < s$, s.t. $f_{-2r} = f_{-2s}\cdot c$ (c – constant). Since the constant terms of f_{-2r} and f_{-2s} coincide, we get $c=1$. However, comparing the leading coeff-s, we get $q^{-2rn} = q^{-2sn} \Rightarrow q^{2(s-r)n} = 1$, where $n := \deg(f)$. As $\substack{n>0 \\ s-r>0}$, $q \neq \sqrt{1}$, we get a contradiction! Thus, E acts nilpotently on V. Same arguments apply to F. ∎

Lemma 3: Assume $\mathrm{char}(k)\neq 2$ (and $q\neq\sqrt{1}$ as before). Let V be a fin. dim. $U_q(sl_2)$-module. Then, $V=\bigoplus_{\lambda\in k^*} V_\lambda$ with $V_\lambda=\{v\in V \mid Kv=\lambda\cdot v\}$. If $V_\lambda\neq 0$, then $\lambda\in\pm q^{\mathbb{Z}}$.

▶ The decomposition $V=\bigoplus_\lambda V_\lambda$ is equivalent to the action of K being diagonalizable. The latter is equivalent to the minimal polynomial of K-action to split into linear factors with multiplicity 1.

By Lemma 2, $\exists N\in\mathbb{N}$ such that $F^N V=0$. Define

$$\boxed{h_r := \prod_{j=-(r-1)}^{r-1} [K;\, r-N+j] \quad \text{for } r\in\mathbb{Z}_{\geq 0}}$$

Exercise 2: Use induction on $0\leq r\leq N$ to prove $F^{N-r} h_r V=0$.

[Hint: The base $r=0$ is due to $h_0=1$, $F^N V=0$.
To complete the induction step, apply Exercise 1 to $E^r F^N \prod_{j=1}^{r-1} [K;\, r-N+j]$.]

In particular, we get $h_N V=0$ (plug $r=N$ into equality of Exercise 2).
However, $h_N=\prod_{j=-(N-1)}^{N-1}[K;j]=\prod_{j=-(N-1)}^{N-1}\frac{Kq^j-K^{-1}q^{-j}}{q-q^{-1}}$ $\Rightarrow$

$$\Rightarrow \left[\prod_{j=-(N-1)}^{N-1}(K-q^{-j})(K+q^{-j})\right](V)=0$$

Thus, the minimal polynomial of K acting on V divides $\prod_{j=-N+1}^{N-1}(K-q^{-j})(K+q^{-j})$, while the latter has simple roots, due to $q\neq\sqrt{1}$. Hence, the result. ■

Question: Can anyone provide a counterexample to Lemma 3 for $\mathrm{char}(k)=2$?
[A: Take $\hat{E}=0$, $\hat{F}=0$, $\hat{K}^{\pm 1}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\in\mathrm{End}(k^2)$.]

Completely analogously to the classical story one can define Verma modules, which inexplicitly can be characterized by the universal property:

U.P. (universal property): For $\lambda\in k^*$, an $U_q(sl_2)$-module V and a vector $v\in V$ s.t. $E(v)=0$, $K(v)=\lambda\cdot v$, there exists a unique $U_q(sl_2)$-morphism $M(\lambda)\longrightarrow V$ s.t. $v_0\mapsto v$
(v_0: distinguished highest weight vector.)

Explicitly, $\boxed{M(\lambda)\simeq U_q(sl_2)/(U_q(sl_2)\cdot E+U_q(sl_2)\cdot(K-\lambda))}$

Applying the PBW theorem for $U_q(sl_2)$, we see that images of $\{F^i\}_{i\geq 0}$ form a basis of $M(\lambda)$. Let $v_i\in M(\lambda)$ denote the image of F^i. The action of $U_q(sl_2)$ in the basis $\{v_i\}_{i\geq 0}$ is following:
(use Lemma 1 for E-action)

$$\boxed{\begin{array}{l} Kv_i=\lambda q^{-2i}v_i, \quad Fv_i=v_{i+1} \\ Ev_i=\begin{cases}0, & i=0\\ [i]\cdot\frac{\lambda q^{1-i}-\lambda^{-1}q^{i-1}}{q-q^{-1}}v_{i-1}, & i>0\end{cases}\end{array}}$$