Rmk: (a) If $q \neq \sqrt{1}$, then $M(\lambda)=\bigoplus_{i\geq 0} M(\lambda)_{q^{-2i}\lambda}$ and $M(\lambda)_{q^{-2i}\lambda}=k v_i$

(b) If q is a primitive d^{th} root of 1, and $e:=\begin{cases} d, & d\text{-odd} \\ d/2, & d\text{-even}\end{cases}$, then

$$M(\lambda)=\bigoplus_{i=0}^{e-1} M(\lambda)_{q^{-2i}\lambda} \text{ and } M(\lambda)_{q^{-2i}\lambda}=\bigoplus_{a\geq 0} k\cdot v_{i+ae}$$

Lemma 1: As before $q\neq\sqrt{1}$ and $\lambda\in k^*$.
(a) If $\lambda\neq\pm q^n$, $n\in\mathbb{Z}_{\geq 0}$, then $M(\lambda)$ is a simple $U_q(\mathfrak{sl}_2)$-module.
(b) If $\lambda=\pm q^n$, $n\in\mathbb{Z}_{\geq 0}$, then $\{v_i\}_{i\geq n+1}$ span an $U_q(\mathfrak{sl}_2)$-submodule of $M(\lambda)$ isomorphic to $M(q^{-2(n+1)}\lambda)$. The latter is the only nontrivial submodule of $M(\lambda)$.

▶ This is based on the standard argument using simplicity of K-spectrum. Let $M'\subset M(\lambda)$ be a nontrivial submodule. Being K-stable $\Rightarrow M'=\bigoplus_\mu M'_\mu$. But $M'_\mu=M'\cap M(\lambda)_\mu$ and all $M(\lambda)_\mu$ are at most 1-dimensional. Thus, M' is spanned by $v_i\in M'$. Let $i_0=\min\{i\in\mathbb{Z}_{\geq 0}\,|\,v_i\in M'\}$. Acting by F, we see $v_{\geq i_0}\in M'$. On the other hand, we must have $E(v_{i_0})=0$. As $i_0>0$ (M'-nontrivial), the latter implies $\lambda q^{1-i_0}-\lambda^{-1}q^{i_0-1}=0 \Rightarrow \boxed{\lambda=\pm q^{i_0-1}\in\pm q^{\mathbb{Z}_{\geq 0}}}$

Part (a) follows.

As for part (b), assume $\lambda=\pm q^n$ ($n\geq 0$). As shown above, there is exactly one nontrivial submodule $M'\subsetneq M(\lambda)$ spanned by $\{v_{n+1},v_{n+2},\dots\}$. As $K(v_{n+1})=\pm q^{-n-2}\cdot v_{n+1}$, $E(v_{n+1})=0$, there must be a $U_q(\mathfrak{sl}_2)$-morphism $M(\pm q^{-n-2})\xrightarrow{\pi} M'$, $v_0\longmapsto v_{n+1}$.
But $\pm q^{-n-2}\notin\pm q^{\mathbb{Z}_{\geq 0}} \overset{(a)}{\Rightarrow} \pi$-injective, while surjectivity is clear.
So: $M'\simeq M(\pm q^{-n-2})$ and it is the only nontrivial submodule of $M(\lambda)$ ■

Prop 1: Assume $q\neq\sqrt{1}$. (a) For every $n\geq 0$, there exist simple modules $L(n,\pm)$ with the bases $\{v_0,v_1,\dots,v_n\}$ (for $L(n,+)$) and $\{v'_0,\dots,v'_n\}$ (for $L(n,-)$) s.t.

$$\boxed{\begin{aligned} &Kv_i=q^{n-2i}v_i,\quad Fv_i=\begin{cases} v_{i+1}, & i<n\\ 0, & i=n\end{cases},\quad Ev_i=\begin{cases}0, & i=0\\ [i][n+1-i]\,v_{i-1}, & i>0\end{cases}\\ &Kv'_i=-q^{n-2i}v'_i,\quad Fv'_i=\begin{cases} v'_{i+1}, & i<n\\ 0, & i=n\end{cases},\quad Ev'_i=\begin{cases}0, & i=0\\ -[i][n+1-i]\cdot v'_{i-1}, & i>0\end{cases}\end{aligned}}$$

(b) Any simple $(n+1)$-dimensional $U_q(\mathfrak{sl}_2)$-module is isomorphic to either of $L(n,\pm)$

▶ (a) Follows from Lemma 1(b): take $L(n,\pm):=M(\pm q^n)/M'$, $M':=\mathrm{span}\langle v_{\geq n+1}\rangle$.
(b) By Lemma 3 from Lecture 6, a fin. dim. $U_q(\mathfrak{sl}_2)$-module V has a weight decomposition $V=\bigoplus_{\lambda\in k^*} V_\lambda$. Since it is finite, $q\neq\sqrt{1}$, $\exists\lambda$ such that $V_\lambda\neq 0$, $V_{q^2\lambda}=0$. As $EV_\lambda\subset V_{q^2\lambda}$, we see that any $v\in V_\lambda\setminus\{0\}$ is a highest weight vector of weight λ.

(Continuation of proof of Prop 1)

By universal property, there is a nonzero $U_q(sl_2)$-morphism $M(\lambda) \xrightarrow{\varphi} V$.
$v_0 \longmapsto v$
As V-simple $\Rightarrow$ φ-surjective $\xRightarrow[\dim V<\infty]{\text{Lemma 1}}$ $\lambda = \pm q^n$, $n \in \mathbb{Z}_{\geq 0}$. But then, again by Lemma 1(b), V is isomorphic to $L(n,\pm)$. ▨

Actually, any finite dimensional $U_q(sl_2)$-module splits into the sum of simple modules, which are of the form $L(n,\pm)$.

Theorem 1: Assume $q \neq \sqrt{1}$, $\mathrm{char}(k) \neq 2$. Then, any finite-dimensional $U_q(sl_2)$-module is semisimple.

One of the standard proofs (borrowed from the classical setting of $U(\mathfrak{g})$) utilizes the quantum Casimir element. We define and discuss the properties of this element first.

Def: The quantum Casimir of $U_q(sl_2)$ is defined as follows:

$$\boxed{C := FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} = EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}}$$

Lemma 2: (a) C is a central element of $U_q(sl_2)$

(b) C acts on each $M(\lambda)$ as a scalar multiplication by $\frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2}$.

▶ (a) $KCK^{-1} = C$ as C has degree 0 (as $\deg(E) = 1$, $\deg(F) = -1$, $\deg(K) = 0$).

The equality $EC = CE$ follows from:

$$EC = EFE + E \cdot \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} = \left(EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}\right) E = CE$$

The equality $FC = CF$ is analogous (or use automorphism $\omega : \omega(C) = C$).

(b) $C(v_0) = \left(FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}\right)(v_0) = \frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2} \cdot v_0$.

As C commutes with $U_q(sl_2)$ by part (a), and v_0 generates $M(\lambda)$, the claim follows. ▨

Lemma 3: (a) C acts on $M(\lambda)$ and $M(\mu)$ by the same scalar iff $\lambda = \mu$ or $\lambda = \mu^{-1}q^{-2}$

(b) If $q \neq \sqrt{1}$, L_1, L_2 – two fin. dim. $U_q(sl_2)$-modules and C acts by the same scalar $\Rightarrow$ $\Rightarrow L_1 \simeq L_2$

▶ (a) Note $\lambda q + \lambda^{-1}q^{-1} = \mu q + \mu^{-1}q^{-1} \Leftrightarrow (\lambda - \mu) q = \frac{(\lambda-\mu)q^{-1}}{\lambda\mu}$. Hence $\lambda = \mu$ or $\lambda = q^{-2}\mu^{-1}$.

(b) As L_1, L_2 are quotients of $M(\varepsilon_1 q^{n_1})$, $M(\varepsilon_2 q^{n_2})$ $\left(\begin{smallmatrix}\varepsilon_1, \varepsilon_2 \in \{\pm\} \\ n_1, n_2 \in \mathbb{Z}_{\geq 0}\end{smallmatrix}\right)$, we can apply part (a) to deduce $\varepsilon_1 q^{n_1} = \varepsilon_2 q^{n_2}$ or $\varepsilon_1 q^{n_1} = \varepsilon_2 q^{-2-n_2}$. In the latter case $q^{2(n_1+n_2+2)} = 1$ contradicting $q \neq \sqrt{1}$. Hence, $\varepsilon_1 q^{n_1} = \varepsilon_2 q^{n_2} \Rightarrow L_1 \simeq L_2$ ▨

Now we can finally prove Theorem 1.

<u>Proof of Theorem 1</u>

► Let V be a finite-dimensional module. Choose any of its composition series $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_m = V$, so that V_i/V_{i-1} is one of the $L(n,\pm)$. Then, C acts by a certain constant λ_i on V_i/V_{i-1}, so that $\boxed{\prod_{i=1}^{m} (C-\lambda_i) V = 0.}$ Therefore, V splits into the direct sum of generalized eigenspaces:

$$\boxed{V = \bigoplus_{\lambda} V_{(\lambda)}, \quad V_{(\lambda)} = \{v \in V \mid (C-\lambda)^s v = 0 \text{ for } s \gg 0\}}$$

As C is central, each $V_{(\lambda)}$ is an $U_q(sl_2)$-submodule. Thus, we can assume from the very beginning that $V = V_{(\lambda)}$ for some $\lambda \in k$. $\Rightarrow$ $C-\lambda$ acts nilpotently on V $\Rightarrow$ also on each V_i/V_{i-1}, whereas the latter action is also by $(\lambda_i - \lambda)\cdot \mathrm{Id}$. Thus $\lambda_i = \lambda$ $\forall i$, so that by Lemma 3, all simple quotients V_i/V_{i-1} are isomorphic, say, to $L(n,\varepsilon)$ ($\varepsilon \in \{\pm\}, n \in \mathbb{Z}_{\geq 0}$). In particular, we get:

(1) $\dim V = m \cdot \dim L(n,\varepsilon)$

(2) $\dim V_\mu = m \cdot \dim L(n,\varepsilon)_\mu$ $\forall \mu$ (these are weight spaces w.r.t. K-action)

As a consequence of (2): $\dim V_{\lambda = \varepsilon q^n} = m$, $\dim V_{q^2\lambda} = 0$. Choose a basis $\{u_1, \ldots, u_m\}$ of V_λ and define $V' := \sum_{i=1}^{m} U_q(sl_2)\cdot u_i$. First, we claim that $V' = V$ as otherwise $(V/V')_\lambda = 0$ while the irreducible subfactors are $\simeq L(n,\varepsilon)$. On the other hand, $\dim\left(\sum_{i=1}^{m} U_q(sl_2)\cdot u_i\right) \leq m \cdot \dim L(n,\varepsilon) = \dim V$ (use u_i - highest weight vector of wt λ)

<u>Thus</u>: $V = V'$ and moreover the sum is direct: $V = \bigoplus U_q(sl_2) u_i$. ■

This completes our discussion of finite-dimensional $U_q(sl_2)$-modules for $q \neq \sqrt{1}$. As we saw, the central el-t C played the central role in the above proof of semisimplicity.

<u>Question</u>: Describe the center of $U_q(sl_2)$.

<u>Theorem 2</u>: If $q \neq \sqrt{1}$, then the center of $U_q(sl_2)$ is generated by C:
$$Z(U_q(sl_2)) \simeq k[C].$$

Let $Z_q := Z(U_q(sl_2))$ be the center of $U_q(sl_2)$.

- First, we note that commutativity with K implies $Z_q \subset U_q(sl_2)_0$ ← degree 0 part of $U_q(sl_2)$ w.r.t. $\mathbb{Z}$-grading from last time. Here we use $q \neq \sqrt{1}$.

Corollary: Any element $z \in Z_q$ can be uniquely written as $z = \sum_{i\geq 0} F^i P_i E^i$, $P_i \in k[K, K^{-1}]$

Consider the projection on U_q^0, $U_q(sl_2)_0 \xrightarrow{\pi} U_q^0$, given by $\sum_{i\geq 0} F^i P_i E^i \longmapsto P_0$.

It is clear that π is an algebra homomorphism whose kernel consists of $\sum_{i\geq 1} F^i P_i E^i$

Def: The homomorphism π is called the Harish-Chandra homomorphism.

Observation: If $z \in Z_q$ and V is a highest weight $U_q(sl_2)$-representation with highest weight λ, then $\boxed{zv = \pi(z)(\lambda) \cdot v}$ (here we evaluate $\pi(z) \in k[K, K^{-1}]$ at λ).

Lemma 4: If $z \in Z_q$ and $\pi(z) = 0$, then $z = 0$.

Assume $z \neq 0$. Then $\pi(z) = 0 \Rightarrow z = \sum_{i=k}^{\ell} F^i P_i E^i$ for some $0 < k \leq \ell$ with $P_k \neq 0$.

Consider a Verma Module $M(\lambda)$ with $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$. Let us apply z to $v_k \in M(\lambda)$.

As $E^{>k} v_k = 0$, while $E^k v_k = c \cdot v_0$ ($c \neq 0$ as $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$), we get $z v_k = c F^k P_k v_0 = c P_k(\lambda) v_k$.

On the other hand, by above observation, $z v_k = \pi(z)(\lambda) v_k = 0$.

Thus: $P_k(\lambda) = 0 \ \forall \lambda \notin \pm q^{\mathbb{Z}_{\geq 0}} \Rightarrow P_k = 0 \Rightarrow$ contradiction (here we assumed $|k| = \infty$).

∎

Using similar arguments with Verma modules we can now prove symmetry of $\widetilde{\pi(z)}$, where given a Laurent polynomial $P \in k[K, K^{-1}]$, set $\widetilde{P}(K) := P(q^{-1}K)$.

Lemma 5: For any element $z \in Z_q$, we have $\widetilde{\pi(z)}(\lambda) = \widetilde{\pi(z)}(\lambda^{-1})$.

According to Lemma 1(b), for every $n \in \mathbb{Z}_{\geq 0}$, we have inclusion of Verma modules $M(q^{-n-2}) \hookrightarrow M(q^n)$. Since both of them are highest weight modules, we must have $\pi(z)(q^n) = \pi(z)(q^{-n-2}) \Rightarrow \widetilde{\pi(z)}(q^n) = \widetilde{\pi(z)}(q^{-n}) \ \forall n \in \mathbb{Z}_{\geq 0}$.

As $q \neq \sqrt{1}$, we get the result

∎

The following is clear:

Lemma 6: If $P \in k[K, K^{-1}]$ satisfies $P(\lambda) = P(\lambda^{-1}) \ \forall \lambda$, then P is a polynomial in $K + K^{-1}$.

Proof of Theorem 2

According to Lemma 4, the homomorphism $Z_q \overset{\pi}{\hookrightarrow} k[K, K^{-1}]$ is injective.

Combining Lemmas 5&6, we also see that its image is contained in $k[qK + q^{-1}K^{-1}]$.

However, due to Lemma 2: $\pi(C) = (qK + q^{-1}K^{-1})/(q - q^{-1})^2$ and $C \in Z_q$.

Thus, C generates Z_q, and moreover $Z_q = Z(U_q(sl_2))$ is a polynomial algebra in C.

∎

• Roots of unity

It is natural to ask what will change if $q=\sqrt{1}$ (a root of unity).

Question 1: What is the center of $U_q(\mathfrak{sl}_2)$ if q is a root of unity?

Question 2: What can be said about fin.dim. $U_q(\mathfrak{sl}_2)$-modules at roots of unity?

Let us first treat the first question.

• Assume q is a primitive d-th root of unity. Define $e:=\begin{cases} d & \text{if } d \text{ is odd} \\ d/2 & \text{if } d \text{ is even}\end{cases}$

Question 1

Lemma 7: The elements E^e, F^e, K^e, K^{-e} are central.

▸ K^e is central as $K^eE=EK^e\cdot q^{2e}=EK^e$, $K^eF=FK^e\cdot q^{-2e}=FK^e$. Same is true about K^{-e}. As for E^e, we only need to check $[F,E^e]=0$. However, the latter follows from Lemma 1(c) of Lecture 6: $[F,E^e]=-[e]\cdot E^{e-1}\cdot[K;e-1]$ and the equality $[e]=0$. As for F^e, one can argue analogously, or immediately deduce from $F^e=\omega(E^e)$ ∎

Lemma 8: Define $U_q(\mathfrak{sl}_2)_0':=\{\sum_{i=0}^{e-1} F^iP_iE^i \mid P_i\in U_q^0\}\subset U_q(\mathfrak{sl}_2)_0$.

(a) The center Z_q is generated by E^e, F^e, and $Z_q\cap U_q(\mathfrak{sl}_2)_0'$.

(b) The restriction of π to $Z_q\cap U_q(\mathfrak{sl}_2)_0'$ is injective.

▸ • Recall the grading $U_q(\mathfrak{sl}_2)=\bigoplus_{k\in\mathbb{Z}} U_q(\mathfrak{sl}_2)_k$, where $\deg(E)=1$, $\deg(F)=-1$, $\deg(K^{\pm1})=0$. As $q^{2i}=1$ iff $i\vdots e$, we see that $Z_q\subset\bigoplus_{k\in\mathbb{Z}} U_q(\mathfrak{sl}_2)_{ke}$. If $k>0$, then $U_q(\mathfrak{sl}_2)_{ke}$ is spanned by $\{F^aK^nE^{a+ke} \mid a\in\mathbb{Z}_{\geq0}, n\in\mathbb{Z}\}$, so that any element $u\in U_q(\mathfrak{sl}_2)_{ke}$ splits as $u=u'\cdot(E^e)^k$, $u'\in U_q(\mathfrak{sl}_2)_0$. Similarly, for $k<0$, any $u\in U_q(\mathfrak{sl}_2)_{ke}$ splits as $u=u'\cdot(F^e)^{-k}$, $u'\in U_q(\mathfrak{sl}_2)_0$.

So: Our first conclusion is that Z_q is generated by E^e, F^e, and $Z_q\cap U_q(\mathfrak{sl}_2)_0$.

• Next, let us derive a condition on $u=\sum_{i\geq0}F^iP_iE^i\in U_q(\mathfrak{sl}_2)_0$ to be central ($P_i\in U_q^0$). The commutativity with K is clear as $\deg(u)=0$. As for E, we use Lemma 1 of Lecture 6:
$$Eu=\sum_{i\geq0}EF^iP_iE^i=\sum_{i\geq0}F^iEP_iE^i+\sum_{i>0}[i]F^{i-1}[K;1-i]P_iE^i=\sum_{i\geq0}F^i\gamma_{-2}(P_i)E^{i+1}+\sum_{i\geq0}[i+1]F^i[K;-i]P_{i+1}E^{i+1}$$
Hence $Eu=uE$ iff $P_i-\gamma_{-2}(P_i)=[i+1]\cdot[K;-i]P_{i+1}$. Same condition is equivalent to $Fu=uF$.

So: u is central if and only if $\boxed{P_i-\gamma_{-2}(P_i)=[i+1]\cdot[K;-i]P_{i+1}\quad \forall i\geq0}$ (♡)

(b) The injectivity of the restriction of π on $Z_q\cap U_q(\mathfrak{sl}_2)_0'$ follows from f-la (♡) together with (1) $[a]\neq0$ for $0<a<e$ and (2) absence of zero divisors in $U_q(\mathfrak{sl}_2)^0$.

(a) To prove (a) in view of above arguments, it suffices to treat $Z_q\cap U_q(\mathfrak{sl}_2)_0$. Any element $u\in Z_q\cap U_q(\mathfrak{sl}_2)_0$ can be written as $u=\sum_{j\geq0}F^{je}\cdot u_j\cdot E^{je}$, where $u_j=\sum_{i=0}^{e-1}F^iP_{je+i}E^i$. Using condition (♡) together with equality $[je]=0$, we see that u-central $\Leftrightarrow$ u_j-central $\forall j$. But $u_j\in Z_q\cap U_q(\mathfrak{sl}_2)_0'$. ∎

Using Lemma 8, it is not very hard to prove the following theorem.

Theorem 3: The center Z_q is generated by E^e, F^e, K^e, K^{-e}, C.

Exercise 1: Prove this theorem.

Question 2

The structure of finite dimensional $U_q(\mathfrak{sl}_2)$-modules at roots of unity is much more interesting (in particular, they are not semisimple anymore).

Modules $W(\lambda, b)$

Let d, e be as before. Choose $\lambda \in k^*$, $b \in k$ and define

$$\boxed{W(\lambda,b) := M(\lambda)/U_q(\mathfrak{sl}_2)\cdot(v_e - b\cdot v_0)}$$

As $[e]=0$, we see that $Ev_e = Ev_0 = 0$. Also $Kv_e = \lambda q^{-2e} v_e = \lambda v_e \Rightarrow K(v_e - bv_0) = \lambda\cdot(v_e - bv_0)$.

So: $U_q(\mathfrak{sl}_2)\cdot(v_e - bv_0)$ is spanned by $F^i(v_e - bv_0) = v_{e+i} - b\cdot v_i$ $(i \in \mathbb{Z}_{\geq 0})$.

In particular, the images of $\{v_0, v_1, \ldots, v_{e-1}\}$ (also denoted by v_i) form a basis of $W(\lambda, b)$.

Explicitly, the $U_q(\mathfrak{sl}_2)$-action on $W(\lambda,b)$ is given by:

$$\boxed{K v_i = \lambda\cdot q^{-2i} v_i, \quad F v_i = \begin{cases} v_{i+1}, & i < e-1 \\ b v_0, & i = e-1 \end{cases}, \quad E v_i = \begin{cases} 0, & i = 0 \\ [i]\cdot\frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q-q^{-1}} v_{i-1}, & i > 0 \end{cases}}$$

Rmks: (a) As $\{1, q^{-2}, q^{-4}, \ldots, q^{-2(e-1)}\}$ are pair-wise distinct, K acts with a simple spectrum.

(b) Note that $F^e = b\cdot \mathrm{Id}_{W(\lambda,b)}$. In particular, Lemma 2 from Lecture 6 is no longer valid.

(c) The last part of Lemma 3 from Lecture 6 is no longer valid as λ - arbitrary.

Lemma 9: (i) If $b \neq 0$ or $\lambda^{2e} \neq 1$, then $W(\lambda,b)$ is a simple $U_q(\mathfrak{sl}_2)$-module.

(ii) If $b=0$ and $\lambda = \pm q^n$ with $0 \leq n < e$, then $W(\lambda,b)$ is simple iff $n = e-1$.

(iii) If $b=0$, $\lambda = \pm q^n$, $0 \leq n < e-1$, then $W(\lambda,b)$ has a unique nontrivial submodule W' spanned by $\{v_{n+1}, v_{n+2}, \ldots, v_{e-1}\}$. Moreover, the quotient $W(\pm q^n, 0)/W'$ is isomorphic to $L(n, \pm)$.

This result is proved similarly to Lemma 1 and is left as an exercise:

Exercise 2: Prove Lemma 9.

Rmk: (a) Note that $W(\pm q^n, 0)$ with $0 \leq n < e-1$ is not semisimple by above result.

(b) Note that E^e acts trivially on $W(\lambda, b)$.

Modules $V(\lambda, a, b)$

Choose $\lambda \in k^*$, $a, b \in k$. Consider a vector space $V(\lambda, a, b)$ spanned by $\{v_0, v_1, \ldots, v_{e-1}\}$.

Define $\boxed{K v_i = \lambda q^{-2i} v_i, \quad F v_i = \begin{cases} v_{i+1}, & i<e-1 \\ b v_0, & i=e-1 \end{cases}, \quad E v_i = \begin{cases} a \cdot v_{e-1}, & i=0 \\ \left(ab + [i] \cdot \frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q-q^{-1}}\right) v_{i-1}, & i>0 \end{cases}}$

It is straightforward to see that this defines an $U_q(\mathfrak{sl}_2)$-module structure, s.t. $E^e = a \cdot Id$, $F^e = b \cdot Id$. Moreover, it is simple if $b \neq 0$.

Exercise 3: (a) Verify that $V(\lambda, a, b)$ is indeed a $U_q(\mathfrak{sl}_2)$-module.
(b) Show that it is simple if $b \neq 0$.

Now we are ready to classify all fin. dim. simple $U_q(\mathfrak{sl}_2)$-modules (assuming k-alg. closed)

- Let V be such a module. As E^e is central, it acts by constant on V (Schur lemma)

Case 1: $E^e_{|V} = 0$.

Then $V' := \{v \in V \mid Ev = 0\}$ is nonzero and is K-stable. Choose an eigenvector $v \in V'$ of K, i.e. $Ev = 0$ and $Kv = \lambda v$. Then, there is an $U_q(\mathfrak{sl}_2)$-morphism $M(\lambda) \xrightarrow{\varphi} V$, $v_0 \mapsto v$. As V is simple, φ-surjective. Let b be the constant by which the central element F^e acts on V. Then $\varphi(v_e - b v_0) = F^e v - b v = 0$ and hence φ factors through $W(\lambda, b)$.

So: By Lemma 9, $\boxed{V \text{ is isomorphic to } W(\lambda, b) \text{ or } L(n, \pm).}$

Case 2: $E^e_{|V} \neq 0$ BUT $F^e_{|V} = 0$.

Recall the Cartan automorphism ω of $U_q(\mathfrak{sl}_2)$. Given a $U_q(\mathfrak{sl}_2)$-module V, we can twist it by ω to get ${}^{\omega}V$. Note that $\omega^2 = Id \Rightarrow {}^{\omega}({}^{\omega}V) \simeq V$. Hence, case 2 reduces to Case 1. Therefore, $\boxed{V \simeq {}^{\omega}W(\lambda, b)}$ (note that $b \neq 0$ by our assumption).

Case 3: $E^e_{|V} = a Id$, $F^e_{|V} = b Id$, and $a, b \neq 0$.

Let $v \in V$ be a K-eigenvector (it exists as k-alg. closed). Let λ be its K-eigenvalue. For $0 \leq i < e$, set $v_i := F^i v$. Note that $F^{e-i} v_i = bv \Rightarrow v_i \neq 0$. Also $K v_i = \lambda q^{-2i} v_i$. As $\{q^{-2i}\}_{0 \leq i < e}$ are pairwise distinct, we see that v_i are lin. indep. Now it is easy to see that $V \simeq V(\lambda, a, b)$.

Thus, we can conclude with the following result:

Theorem 4: Any simple finite-dimensional $U_q(\mathfrak{sl}_2)$-module is one of the following: $\{L(n, \pm)\}_{0 \leq n < e-1} \cup \{W(\lambda, b) \mid b \neq 0 \text{ or } \lambda^{2e} \neq 1\} \cup \{{}^{\omega}W(\lambda, b) \mid b \neq 0\} \cup \{V(\lambda, a, b) \mid {}^{a \neq 0}_{b \neq 0}\}$.