During the last two lectures we studied the category of fin. dim. repr-s of $U_q(\mathfrak{sl}_2)$, as well as its algebra properties, but we didn't show yet it is a Hopf alg.

Set:

$$\Delta(E) = E\otimes 1 + K\otimes E\,, \quad \Delta(F) = F\otimes K^{-1} + 1\otimes F\,, \quad \Delta(K) = K\otimes K \qquad (1)$$

$$S(E) = -K^{-1}E\,, \quad S(F) = -FK\,, \quad S(K) = K^{-1} \qquad (2)$$

$$\varepsilon(E) = 0\,, \quad \varepsilon(F) = 0\,, \quad \varepsilon(K) = 1 \qquad (3)$$

Theorem 1: $U_q(\mathfrak{sl}_2)$ has a Hopf algebra structure with coproduct Δ determined by (1), antipode S determined by (2), counit ε - by (3).

- First, we verify that (1) determine an alg. homomorphism $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)$. In other words, one needs to verify that 4 elements of $U_q(\mathfrak{sl}_2)^{\otimes 2}$
$$E\otimes 1 + K\otimes E\,, \quad F\otimes K^{-1} + 1\otimes F\,, \quad K\otimes K,\ K^{-1}\otimes K^{-1}$$
satisfy the defining 4 rel-s of $U_q(\mathfrak{sl}_2)$. The first 3 rel-s are immediate as $\deg(E)=1$, $\deg(K)=0$, $\deg(F)=-1$. It remains to check the last one:
$$[E\otimes 1 + K\otimes E,\ F\otimes K^{-1} + 1\otimes F] = [E,F]\otimes K^{-1} + K\otimes [E,F] + KF\otimes EK^{-1} - FK\otimes K^{-1}E =$$
$$= \frac{K-K^{-1}}{q-q^{-1}}\otimes K^{-1} + K\otimes \frac{K-K^{-1}}{q-q^{-1}} + q^{-2}FK\otimes EK^{-1} - q^{-2}FK\otimes EK^{-1} = \frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}} \quad \checkmark$$

- Coassociativity of Δ is an easy check.

- Next, we check that f-las (3) define an algebra homom. $U_q(\mathfrak{sl}_2) \to k$. For this one needs to verify $(0,0,1,1)$ satisfy the 4 rel-s of $U_q(\mathfrak{sl}_2)$ which is obvious.

- The counity property is an immediate check, i.e. the commutativity of
$$\begin{tikzcd} U_q(\mathfrak{sl}_2) \arrow[r, "\Delta"] \arrow[dr, "\simeq"'] & U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2) \arrow[d, "\mathrm{Id}\otimes\varepsilon"] \\ & U_q(\mathfrak{sl}_2)\otimes k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U_q(\mathfrak{sl}_2) \arrow[r, "\Delta"] \arrow[dr, "\simeq"'] & U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2) \arrow[d, "\varepsilon\otimes\mathrm{Id}"] \\ & k\otimes U_q(\mathfrak{sl}_2) \end{tikzcd}$$

- Next, we check that f-las (2) define an algebra homom. $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{op}$. In other words, one needs to verify that the 4 el-s of $U_q(\mathfrak{sl}_2)^{op}$:
$$-K^{-1}E,\ -FK,\ K^{-1},\ K$$
satisfy the defining 4 rel-s of $U_q(\mathfrak{sl}_2)$.

The verification of the first 3 is immediate. It remains to treat the last:

$(-K^{-1}E)\bullet(-FK)-(-FK)\bullet(-K^{-1}E) = FE - q^{-2}\cdot q^{2}\cdot EF = -\frac{K-K^{-1}}{q-q^{-1}} = \frac{K^{-1}-K}{q-q^{-1}}$, ✓

where "$\bullet$" was used for the product in $U_q(\mathfrak{sl}_2)^{op}$

• Finally, one needs to check $\sum_{(x)} S(x')x'' = \sum_{(x)} x'S(x'') = \eta\varepsilon(x)$.

As we know from Lecture #2 (Lemma 2 of), it suffices to verify these equalities for the 4 generators of $U_q(\mathfrak{sl}_2)$.

∘ $\sum_{(E)} S(E')\cdot E'' = -K^{-1}E + K^{-1}E = 0$, $\sum_{(E)} E'\cdot S(E'') = E - K\cdot K^{-1}E = 0$.

∘ $\sum_{(F)} S(F')\cdot F'' = -FK\cdot K^{-1} + 1\cdot F = 0$, $\sum_{(F)} F'\cdot S(F'') = FK - FK = 0$

∘ $\sum_{(K^{\pm 1})} S((K^{\pm1})')(K^{\pm1})'' = K^{\mp 1}\cdot K^{\pm 1} = 1 = \sum_{(K^{\pm1})} (K^{\pm1})'\cdot S((K^{\pm1})'')$

■

<u>Lemma 1</u>: (a) $S^2(u) = K^{-1}uK \quad \forall u \in U_q(\mathfrak{sl}_2)$

(b) $S(E^r) = (-1)^r q^{r(r-1)} K^{-r}E^r$, $S(F^r) = (-1)^r q^{-r(r-1)} F^r K^r$

► (a) $S^2(K) = K$, $S^2(E) = S(-K^{-1}E) = -(-K^{-1}E)K = K^{-1}EK$, $S^2(F) = S(-FK) = -K^{-1}(-FK) = K^{-1}FK$.

Both S^2 and $u \mapsto K^{-1}uK$ are algebra homom. $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$, coinciding on generators $K^{\pm1}, E, F$ $\Rightarrow$ they are equal.

(b) Prove by induction. Case $r=1$ is just f-la (2).

$S(E^{r+1}) = S(E\cdot E^r) = S(E^r)S(E) = (-1)^r q^{r(r-1)}K^{-r}E^r\cdot(-1)K^{-1}E = (-1)^{r+1} q^{\overbrace{r(r-1)+2r}^{r(r+1)}} K^{-r-1}E^{r+1}$

$S(F^{r+1}) = S(F\cdot F^r) = S(F^r)S(F) = (-1)^r q^{-r(r-1)} F^r K^r\cdot(-1)FK = (-1)^{r+1} q^{-r(r-1)-2r} F^{r+1}K^{r+1}$

completing the induction step.

■

<u>Lemma 2</u>: $\Delta(K^n) = K^n \otimes K^n$, $\Delta(E^r) = \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^{r-i}K^i \otimes E^i$, $\Delta(F^r) = \sum_{i=0}^{r} q^{i(r-i)}\begin{bmatrix} r \\ i \end{bmatrix} F^i \otimes F^{r-i}K^i$

► • The first equality is obvious.

• $\Delta(E^r) = (E\otimes 1 + K\otimes E)^r \underset{\left(\substack{K\otimes E\,\cdot\, E\otimes 1 = \\ = q^{2} E\otimes 1\,\cdot\, K\otimes E}\right)}{=\!=\!=} \sum_{i=0}^{r} \binom{r}{r-i}_{q^2} E^{r-i}K^i\otimes E^i = \sum_{i=0}^{r} q^{i(r-i)}\begin{bmatrix} r \\ i \end{bmatrix} E^{r-i}K^i \otimes E^i$

• $\Delta(F^r) = (F\otimes K^{-1} + 1\otimes F)^r = \sum_{i=0}^{r}\binom{r}{r-i}_{q^2}(1\otimes F)^{r-i}(F\otimes K^{-1})^i = \sum_{i=0}^{r} q^{i(r-i)}\begin{bmatrix} r \\ i \end{bmatrix} F^i \otimes F^{r-i}K^{-i}$

■

<u>Rmk</u>: Recall the algebra $\widetilde{U}_q(\mathfrak{sl}_2)$ of Lecture 5, which was introduced to relate $q=1$ limit of $U_q(\mathfrak{sl}_2)$ to $U(\mathfrak{sl}_2)$: $\widetilde{U}_1(\mathfrak{sl}_2)/(K-1) \simeq U(\mathfrak{sl}_2)$.

Then, $\widetilde{U}_q(\mathfrak{sl}_2)$ is a Hopf alg. with f-las (1-3) and $\Delta(L) = L\otimes K + K^{-1}\otimes L$, $\varepsilon(L) = 0$, $S(L) = -L$.

Moreover, the isomorphism $\widetilde{U}_1(\mathfrak{sl}_2)/(K-1) \simeq U(\mathfrak{sl}_2)$ is an isom. of Hopf alg-s.

For now on, assume $q \neq \sqrt{1}$ and $\mathrm{char}(k) \neq 2$.

Define $\Theta_{-1} := 0$, $\Theta_0 := 1 \otimes 1$, $\Theta_n := a_n \cdot F^n \otimes E^n \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, where the coefficients $\{a_n\}_{n=0}^{\infty}$ satisfy $a_0 = 1$, $a_n = -q^{-(n-1)} \cdot \frac{q-q^{-1}}{[n]} a_{n-1} \Rightarrow \boxed{a_n = (-1)^n q^{-\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!}}$

Lemma 3: For any $n \in \mathbb{Z}_{\geq 0}$, we have:

$$(K \otimes K)\Theta_n = \Theta_n (K \otimes K)$$
$$(E \otimes 1)\Theta_n + (K \otimes E)\Theta_{n-1} = \Theta_n (E \otimes 1) + \Theta_{n-1}(K^{-1} \otimes E)$$
$$(1 \otimes F)\Theta_n + (F \otimes K^{-1})\Theta_{n-1} = \Theta_n (1 \otimes F) + \Theta_{n-1}(F \otimes K)$$

▸ • The first equality is obvious.

• $[E \otimes 1, \Theta_n] = [E \otimes 1, a_n F^n \otimes E^n] = [n] \cdot a_n \cdot F^{n-1}[K; 1-n] \otimes E^n$

$\Theta_{n-1}(K^{-1} \otimes E) - (K \otimes E)\Theta_{n-1} = a_{n-1}(F^{n-1}K^{-1} \otimes E^n - KF^{n-1} \otimes E^n) = a_{n-1} \cdot F^{n-1}(K^{-1} - K \cdot q^{-2(n-1)}) \otimes E^n$

and these two expressions coincide, due to a recursive f-la on $\{a_n\}$.

• The third equality is analogous ■

Recall that under our assumptions on q, k, any f.d. $U_q(\mathfrak{sl}_2)$-repr. has a weight decomposition w.r.t. K-action. Moreover, the set of all possible weights is

$$\boxed{\Lambda := \{\pm q^a \mid a \in \mathbb{Z}\}.}$$

Pick a map $f: \Lambda \times \Lambda \longrightarrow k^\times$ s.t.

$$\boxed{f(\lambda, \mu) = \lambda f(\lambda, \mu q^2) = \mu f(\lambda q^2, \mu) \quad \forall \lambda, \mu \in \Lambda.}$$

Exercise 1: Classify all such maps f.

For any two f.d. $U_q(\mathfrak{sl}_2)$-modules M_1, M_2, consider a corresponding linear map (isom. of vector spaces) $\boxed{\mathcal{F}: M_1 \otimes M_2 \to M_1 \otimes M_2 \text{ given by } \mathcal{F}(m_1 \otimes m_2) = f(\lambda, \mu) \cdot m_1 \otimes m_2}$ for any $m_1 \in (M_1)_\lambda$, $m_2 \in (M_2)_\mu$.

Also recall that E, F act nilpotently on M_1, M_2, hence the sum $\Theta := \sum_{n \geq 0} \Theta_n : M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2$ is well-defined.

Finally, we set

$$\boxed{\Theta^f := \Theta \circ \mathcal{F} : M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2}$$

Let $\tau = \tau_{M_2, M_1} : M_2 \otimes M_1 \longrightarrow M_1 \otimes M_2$ be the "swap" map

Theorem 2: The map $\textcircled{H}^{f} \circ \tau : M_2 \otimes M_1 \longrightarrow M_1 \otimes M_2$ is an isomorphism of $U_q(\mathfrak{sl}_2)$-modules.

▶ • First, we note that τ is bijective. As $F \otimes E$ acts nilpotently on $M_1 \otimes M_2$, the operator $\textcircled{H}$ is also (as $\textcircled{H}_0 = \mathbb{1}$) bijective $\Rightarrow \textcircled{H}^f$ is bijective. Thus, $\textcircled{H}^f \circ \tau$ – bijective.

• Second, note that for any $x \in U_q(\mathfrak{sl}_2)$, $w \in M_2 \otimes M_1$, we have:

$$\tau(x(w)) = \tau(\Delta(x) w) = \Delta^{op}(x)(\tau(w)).$$

For that reason, $\textcircled{H}^f \circ \tau$ intertwines $U_q(\mathfrak{sl}_2)$-actions iff $\forall x \in U_q(\mathfrak{sl}_2)$:

$$\boxed{\Delta(x) \circ \textcircled{H}^f = \textcircled{H}^f \circ \Delta^{op}(x)} \quad (*)$$

It suffices to verify $(*)$ on the generators $x = E, F, K^{\pm 1}$.

• Recalling Lemma 3, we see that

$\Delta(K)\textcircled{H} = \textcircled{H}\,\tilde{\Delta}(K)$, $\Delta(E)\textcircled{H} = \textcircled{H}\,\tilde{\Delta}(E)$, $\Delta(F)\textcircled{H} = \textcircled{H}\,\tilde{\Delta}(F)$, where

$\tilde{\Delta}(K) = K \otimes K$, $\tilde{\Delta}(E) = E \otimes 1 + K^{-1} \otimes E$, $\tilde{\Delta}(F) = 1 \otimes F + F \otimes K$.

Therefore, it suffices to check $\tilde{\Delta}(x) \circ \tilde{f} = \tilde{f} \circ \Delta^{op}(x)$ for generators, i.e.:

$$(K \otimes K) \circ \tilde{f} = \tilde{f} \circ (K \otimes K)$$
$$(E \otimes 1 + K^{-1} \otimes E) \circ \tilde{f} = \tilde{f}(E \otimes K + 1 \otimes E)$$
$$(1 \otimes F + F \otimes K) \circ \tilde{f} = \tilde{f}(F \otimes 1 + K^{-1} \otimes F)$$

Pick $m_1 \in (M_1)_\lambda$, $m_2 \in (M_2)_\mu$. As $Km_1 \in (M_1)_\lambda$, $Km_2 \in (M_2)_\mu$, the first equality follows.

As for the second equality:

$$(E \otimes 1 + K^{-1} \otimes E) \circ \tilde{f}(m_1 \otimes m_2) = f(\lambda, \mu) Em_1 \otimes m_2 + \lambda^{-1} f(\lambda, \mu) \cdot m_1 \otimes Em_2$$
$$\tilde{f} \circ (E \otimes K + 1 \otimes E)(m_1 \otimes m_2) = f(q^2\lambda, \mu) \cdot \mu \cdot Em_1 \otimes m_2 + f(\lambda, q^2\mu)\, m_1 \otimes Em_2$$

which coincide due to conditions on f.

As for the third equality:

$$(1 \otimes F + F \otimes K) \circ \tilde{f}(m_1 \otimes m_2) = f(\lambda, \mu) \cdot m_1 \otimes Fm_2 + f(\lambda, \mu) \cdot \mu \cdot Fm_1 \otimes m_2$$
$$\tilde{f} \circ (F \otimes 1 + K^{-1} \otimes F)(m_1 \otimes m_2) = f(q^{-2}\lambda, \mu) \cdot Fm_1 \otimes m_2 + f(\lambda, q^{-2}\mu) \cdot \lambda^{-1} \cdot m_1 \otimes Fm_2$$

which again coincide, due to our choice of f ∎

Rmk: Actually, $\tilde{\Delta}(x) = (\sigma \otimes \sigma)(\Delta(\sigma^{-1}(x)))$, where σ is an antiautomorphism of $U_q(\mathfrak{sl}_2)$ s.t. $\sigma: E \mapsto E$, $F \mapsto F$, $K^{\pm 1} \mapsto K^{\mp 1}$.

Rmk: Note that the fact that f.d. modules $M_1\otimes M_2$ and $M_2\otimes M_1$ are apriori isomorphic as $U_q(\mathfrak{sl}_2)$-modules follows from the weight decomposition of both (as we know that M_1, M_2 have weight decompositions w.r.t. K-action, $\Delta(K)=K\otimes K$, and semisimplicity of f.d. $U_q(\mathfrak{sl}_2)$-modules together with explicit description of weight for each simple $L(n,\pm)$)

However, Theorem 2 provides a functorial isomorphism.
In other words, if $M_1 \xrightarrow{f_1} N_1$, $M_2 \xrightarrow{f_2} N_2$ are $U_q(\mathfrak{sl}_2)$-morphisms, then

$$\begin{tikzcd} M_2\otimes M_1 \arrow[r, "\Theta^f\circ\tau"] \arrow[d, "f_2\otimes f_1"] & M_1\otimes M_2 \arrow[d, "f_1\otimes f_2"] \\ N_2\otimes N_1 \arrow[r, "\Theta^f\circ\tau"] & N_1\otimes N_2 \end{tikzcd}$$

is commutative.

Rmk: In the above discussion M_1, M_2 were assumed to be fin.dim.
On the other hand, if this was true for any M_1, M_2, then the above functoriality would imply that all these isomorphisms are induced from a single one for $M_1\simeq M_2\simeq U_q(\mathfrak{sl}_2)$. In the latter case, the analogue of $(*)$ reads as $\Delta(x)\mathcal{R}=\mathcal{R}\Delta^{op}(x)$, where $\mathcal{R}\in U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)$.
In the numeric setting we have been working in ($q\in k^*$), there is no such $\mathcal{R}$, which forces us to restrict consideration only to f.dim. repr-s.
However, when working in the formal setting (over $k[[\hbar]]$, $q=e^{\hbar/2}$) such $\mathcal{R}$ exists (and was found by Drinfeld). I hope we will get to this point later in the class.

Let us conclude with an important property of the isomorphisms of Theorem 2. Pick three f.dim. $U_q(\mathfrak{sl}_2)$-modules M_1, M_2, M_3. Then, we have three automorphisms of $M_1\otimes M_2\otimes M_3$: $\Theta^f_{12}:=\Theta^f\otimes 1$, $\Theta^f_{23}:=1\otimes\Theta^f$, and $\Theta^f_{13}:=(1\otimes\tau_{M_3,M_2})\circ(\Theta^f\otimes 1)\circ(1\otimes\tau_{M_2,M_3})$.

Theorem 3: We have $\Theta^f_{12}\circ\Theta^f_{13}\circ\Theta^f_{23}=\Theta^f_{23}\circ\Theta^f_{13}\circ\Theta^f_{12}$

Corollary: In the case $M_1=M_2=M_3=M$, $R:=\Theta^f\in \mathrm{End}(M\otimes M)$ satisfies the quantum Yang-Baxter eq-n $\boxed{R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}}$

▸ (Proof of Theorem 3)

$\text{LHS} = \Theta_{12}\circ \tilde{f}_{12}\circ \Theta_{13}\circ \tilde{f}_{13}\circ \Theta_{23}\circ \tilde{f}_{23}$

$\text{RHS} = \Theta_{23}\circ \tilde{f}_{23}\circ \Theta_{13}\circ \tilde{f}_{13}\circ \Theta_{12}\circ \tilde{f}_{12}$

First, we shall rewrite both LHS & RHS by taking Θ's to the left of $\tilde{f}$'s.

First, we define $\Theta'_n, \Theta''_n \in U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)$ via

$$\boxed{\Theta'_n := a_n\cdot F^n\otimes K^n\otimes E^n, \quad \Theta''_n := a_n\cdot F^n\otimes K^{-n}\otimes E^n.}$$

and we set

$$\boxed{\Theta' := \sum_{n\geq 0}\Theta'_n, \quad \Theta'' := \sum_{n\geq 0}\Theta''_n}$$

Lemma 4: (a) $\tilde{f}_{12}\circ\Theta_{13} = \Theta'\circ\tilde{f}_{12}$

(b) $\tilde{f}_{23}\circ\Theta_{13} = \Theta''\circ\tilde{f}_{23}$

▸ (a) Pick $w = m_1\otimes m_2\otimes m_3 \in (M_1)_\lambda\otimes(M_2)_\mu\otimes(M_3)_\nu$. Then:

$$\tilde{f}_{12}\circ\Theta_{13}(w) = \sum_{n\geq 0} a_n\cdot f(\lambda q^{-2n},\mu)\cdot F^n m_1\otimes m_2\otimes E^n m_3 = \sum_{n\geq 0} a_n f(\lambda,\mu)\mu^n F^n m_1\otimes m_2\otimes E^n m_3$$

$$= \sum_{n\geq 0} f(\lambda,\mu)\cdot a_n\cdot F^n m_1\otimes K^n m_2\otimes E^n m_3 = \Theta'\circ\tilde{f}_{12}(w) \quad \checkmark$$

(b) $$\tilde{f}_{23}\circ\Theta_{13}(w) = \sum_{n\geq 0} a_n f(\mu,\nu q^{2n})\cdot F^n m_1\otimes m_2\otimes E^n m_3 = \sum_{n\geq 0} a_n f(\mu,\nu)\mu^{-n} F^n m_1\otimes m_2\otimes E^n m_3$$

$$= \sum_{n\geq 0} f(\mu,\nu)\cdot a_n\cdot F^n m_1\otimes K^{-n} m_2\otimes E^n m_3 = \Theta''\circ\tilde{f}_{23}(w) \quad \checkmark$$

∎

Lemma 5: (a) $\tilde{f}_{12}\circ\tilde{f}_{13}\circ\Theta_{23} = \Theta_{23}\circ\tilde{f}_{12}\circ\tilde{f}_{13}$

(b) $\tilde{f}_{23}\circ\tilde{f}_{13}\circ\Theta_{12} = \Theta_{12}\circ\tilde{f}_{23}\circ\tilde{f}_{13}$

▸ We will verify (a) following Lemma 4, part (b) is similar.

$$\tilde{f}_{12}\circ\tilde{f}_{13}\circ\Theta_{23}(w) = \sum_{n\geq 0} a_n f(\lambda,\mu q^{-2n}) f(\lambda,\nu q^{2n})\cdot m_1\otimes F^n m_2\otimes E^n m_3 =$$

$$= \sum_{n\geq 0} a_n\cdot f(\lambda,\mu)\cdot\cancel{\lambda^n}\cdot f(\lambda,\nu)\cdot\cancel{\lambda^{-n}}\cdot m_1\otimes F^n m_2\otimes E^n m_3 = \Theta_{23}\circ\tilde{f}_{12}\circ\tilde{f}_{13}(w)$$

∎

Using Lemmas 4–5, we see that Theorem 3 boils down to the proof of

$$\boxed{\Theta_{12}\circ\Theta'\circ\Theta_{23} = \Theta_{23}\circ\Theta''\circ\Theta_{12}} \quad (\dagger)$$

Exercise 2 (next time Oleksii will present the proof): Prove $(\dagger)$.

∎