Last time: • Hopf algebra structure on $U_q(\mathfrak{sl}_2)$

• For any two f. dimensional $U_q(\mathfrak{sl}_2)$-modules, we constructed a functorial isomorphism $M_2\otimes M_1 \xrightarrow{\sim} M_1\otimes M_2$ (of $U_q(\mathfrak{sl}_2)$-modules) in the form $\Theta^f \circ \tau$ (τ – permutation $M_2\otimes M_1 \longrightarrow M_1\otimes M_2 : m_2\otimes m_1 \longmapsto m_1\otimes m_2$).

Rmk 1: I forgot to mention last time that this Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ deforms the canonical Hopf algebra structure on $U(\mathfrak{sl}_2)$. To state the result, recall the algebra $\widetilde{U}_q(\mathfrak{sl}_2)$ from Lecture 5, which was generated by $\{E, F, K^{\pm 1}, L\}$ and isomorphic to $U_q(\mathfrak{sl}_2)$ for $q\neq \pm 1$, while specialization $\widetilde{U}_{q=1}(\mathfrak{sl}_2)$ was well-defined and $U(\mathfrak{sl}_2)\simeq \widetilde{U}_{q=1}(\mathfrak{sl}_2)/(K-1)$ as algebras.

The point now is that $\widetilde{U}_q(\mathfrak{sl}_2)$ has a Hopf alg. structure, s.t. the isomorphism $U(\mathfrak{sl}_2)\simeq \widetilde{U}_{q=1}(\mathfrak{sl}_2)/(K-1)$ is an isom. of Hopf alg-s. The images of $E, F, K^{\pm 1}$ under Δ, ε, S are defined as in $U_q(\mathfrak{sl}_2)$, while we set $\Delta(L) := L\otimes K + K^{-1}\otimes L$, $\varepsilon(L) := 0$, $S(L) := -L$. (Exercise 1: Check the claim).

Rmk 2: In the simplest case of $M_1 = M_2 = L(1,+)$ and a map $f : \hat{\Lambda}\times\hat{\Lambda} \to k$ as in Lecture 8, verify that Θ^f is given by the following 4×4 matrix

$$\Theta^f = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda q & 0 & 0 \\ 0 & \lambda q(q^{-1}-q) & \lambda q & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda := f(q,q)$ and the basis we use is $\{m_0\otimes m_0,\ m_0\otimes m_1,\ m_1\otimes m_0,\ m_1\otimes m_1\}$.

Rmk 3: Last time we played with three coproducts:

(1) Δ: $K\longmapsto K\otimes K$, $E\longmapsto E\otimes 1 + K\otimes E$, $F\longmapsto F\otimes K^{-1} + 1\otimes F$

(2) Δ^{op}: $K\longmapsto K\otimes K$, $E\longmapsto E\otimes K + 1\otimes E$, $F\longmapsto F\otimes 1 + K^{-1}\otimes F$

(3) $\widehat{\Delta}$: $K\longmapsto K\otimes K$, $E\longmapsto E\otimes 1 + K^{-1}\otimes E$, $F\longmapsto F\otimes K + 1\otimes F$

it is related to Δ via the following formula $\widehat{\Delta}(x) = (\sigma\otimes\sigma)\Delta(\sigma^{-1}(x))$, where σ is an anti-involution of $U_q(\mathfrak{sl}_2)$ given by $E\mapsto E$, $F\mapsto F$, $K\mapsto K^{-1}$.

During the proof of Thm 2 of Lecture 8, we verified

$$\left.\begin{array}{l} \Theta\,\widehat{\Delta}(x) = \Delta(x)\,\Theta \\ \widehat{\Delta}(x)\,\mathcal{F} = \mathcal{F}\,\Delta^{op}(x) \end{array}\right\} \Rightarrow \Delta(x)\,\Theta^f = \Theta^f\,\Delta^{op}(x),$$ which is equivalent to Thm 2.

Last time, we concluded with equality b/w $\Theta, \Theta', \Theta''$, denoted $(\dagger)$ in Lecture 8. Let us now provide a proof for it. We recall that $\Theta' = \sum_{n\geq 0} \Theta'_n$, $\Theta'' = \sum_{n\geq 0} \Theta''_n$ with $\Theta'_n = a_n F^n \otimes K^n \otimes E^n$, $\Theta''_n = a_n F^n \otimes K^{-n} \otimes E^n$

Prop 1: We have

$$\boxed{\Theta_{12} \circ \Theta' \circ \Theta_{23} = \Theta_{23} \circ \Theta'' \circ \Theta_{12}} \quad (\dagger)$$

▸ • $\Theta_{23} \circ \Theta'' = \sum_{i,j\geq 0} a_i a_j (1 \otimes F^j \otimes E^j)(F^i \otimes K^{-i} \otimes E^i) = \sum_{n\geq 0} \sum_{i=0}^{n} a_i a_{n-i} \cdot F^i \otimes F^{n-i} K^{-i} \otimes E^i$

On the other hand, due to [Lecture 8, Lemma 2], we have:

$\Delta(F^n) = \sum_{i=0}^{n} q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} F^i \otimes F^{n-i} K^{-i}$. However, we have:

$a_i a_{n-i} = (-1)^n \cdot q^{-\frac{i(i-1)}{2} - \frac{(n-i)(n-i-1)}{2}} \cdot \frac{(q-q^{-1})^n}{[i]!\,[n-i]!} = a_n \cdot \begin{bmatrix} n \\ i \end{bmatrix} \cdot q^{\frac{n(n-1) - i(i-1) - (n-i)(n-i-1)}{2}} = a_n \cdot q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}$ $\Rightarrow$

$\Rightarrow \boxed{\Theta_{23} \circ \Theta'' = (\Delta \otimes 1)(\Theta)}$

• Last time (see also Rmk 3 above), we proved $\Delta(x)\Theta = \Theta \Delta^{op}(x)$. $\Rightarrow$

$\Rightarrow$ Right-hand side of $(\dagger)$ equals $\boxed{\Theta_{12} \circ (\tilde{\Delta} \otimes 1)(\Theta)}$

• Hence, to prove $(\dagger)$, it suffices to verify $(\tilde{\Delta} \otimes 1)(\Theta) = \Theta' \circ \Theta_{23}$.

But $\Theta' \circ \Theta_{23}$ can be computed as $\Theta_{23} \circ \Theta''$ above, that is:

$\Theta' \circ \Theta_{23} = \sum_{i,j\geq 0} a_i a_j (F^i \otimes K^i \otimes E^i)(1 \otimes F^j \otimes E^j) = \sum_{n\geq 0} \sum_{i=0}^{n} a_i a_{n-i} \cdot F^i \otimes K^i F^{n-i} \otimes E^n$

But $\tilde{\Delta}(F) = F \otimes K + 1 \otimes F \overset{q\text{-binomial}}{\Longrightarrow} \tilde{\Delta}(F^n) = \sum_{i=0}^{n} q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} F^i \otimes K^i F^{n-i}$

$a_i a_{n-i} = a_n \cdot q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}$ (above) $\Rightarrow$

$\Rightarrow \boxed{\Theta' \circ \Theta_{23} = (\tilde{\Delta} \otimes 1)(\Theta)}$

This completes the proof of Prop 1, hence, completing the proof of Thm 2 from Lecture 8.

Recall that our construction of intertwiners $\Theta f \circ \tau$ depends on a choice of a map $f: \Lambda \times \Lambda \longrightarrow k^*$ (here $\Lambda = \{\pm q^{\mathbb{Z}}\}$) subject to:

$$\boxed{f(\lambda, \mu) = \lambda \cdot f(\lambda, \mu q^2) = \mu \cdot f(\lambda q^2, \mu)} \quad (1)$$

There is a plenty of such maps, but requesting some additional properties, restricts the choice - see our next result.

Prop 2: Let M_1, M_2, M_3 be three finite dimensional $U_q(\mathfrak{sl}_2)$-modules. We also assume that the map f satisfying (1) also satisfies

$$\boxed{f(\lambda,\mu\nu)=f(\lambda,\nu)\,f(\lambda,\mu)\,,\quad f(\lambda\mu,\nu)=f(\lambda,\nu)\,f(\mu,\nu)} \qquad (2)$$

for all weights λ,μ,ν of these modules. Then, the following diagrams commute:

$$(\diamondsuit_1)\qquad \begin{tikzcd}[column sep=small]
 & M_1\otimes(M_3\otimes M_2) \arrow[r, "can"] & (M_1\otimes M_3)\otimes M_2 \arrow[dr, "(\check{R}\otimes 1)"] & \\
M_1\otimes(M_2\otimes M_3) \arrow[ur, "1\otimes\check{R}"] \arrow[dr, "can"'] & & & (M_3\otimes M_1)\otimes M_2 \\
 & (M_1\otimes M_2)\otimes M_3 \arrow[r, "\check{R}"] & M_3\otimes(M_1\otimes M_2) \arrow[ur, "can"'] &
\end{tikzcd}$$

and

$$(\diamondsuit_2)\qquad \begin{tikzcd}[column sep=small]
 & (M_2\otimes M_1)\otimes M_3 \arrow[r, "can"] & M_2\otimes(M_1\otimes M_3) \arrow[dr, "1\otimes\check{R}"] & \\
(M_1\otimes M_2)\otimes M_3 \arrow[ur, "\check{R}\otimes 1"] \arrow[dr, "can"'] & & & M_2\otimes(M_3\otimes M_1) \\
 & M_1\otimes(M_2\otimes M_3) \arrow[r, "\check{R}"] & (M_2\otimes M_3)\otimes M_1 \arrow[ur, "can"'] &
\end{tikzcd}$$

Here can: $A\otimes(B\otimes C)\xrightarrow{\sim}(A\otimes B)\otimes C$ via $a\otimes(b\otimes c)\mapsto(a\otimes b)\otimes c$, while $\check{R}:=\Theta^f\circ\tau : B\otimes A\to A\otimes B$ for any $U_q(\mathfrak{sl}_2)$-modules A,B,C.

▶ "Upper half" of the diagram $(\diamondsuit_1)$ equals

$$\Theta_{12}\circ\tilde{f}_{12}\circ\tau_{12}\circ\Theta_{23}\circ\tilde{f}_{23}\circ\tau_{23}=\Theta_{12}\circ\tilde{f}_{12}\circ\Theta_{13}\circ\tau_{12}\circ\tilde{f}_{23}\circ\tau_{23}=\Theta_{12}\circ\tilde{f}_{12}\circ\Theta_{13}\circ\tilde{f}_{13}\circ\tau_{12}\circ\tau_{23}$$

$$\underset{\text{Lemma 4 of Lect 8}}{=}\Theta_{12}\circ\Theta'\circ\tilde{f}_{12}\circ\tilde{f}_{13}\circ\tau_{12}\circ\tau_{23}$$

"Lower half" of the diagram $(\diamondsuit_1)$ equals

$(1\otimes\Delta)(\Theta)\circ\tilde{f}'\circ\tau_{12}\circ\tau_{23}$, where $\tilde{f}'(w)=f(\lambda,\mu\nu)w$ for $w\in(M_3)_\lambda\otimes(M_1)_\mu\otimes(M_2)_\nu$.

But: analogously to our computations in the proof of Prop 2, one can check that $\boxed{(1\otimes\Delta)(\Theta)=\Theta_{12}\circ\Theta'}$

Thus, the commutativity of $(\diamondsuit_1)$ follows from $f(\lambda,\mu\nu)=f(\lambda,\mu)f(\lambda,\nu)$.

Likewise, the commutativity of $(\diamondsuit_2)$ follows from $f(\lambda\mu,\nu)=f(\lambda,\nu)f(\mu,\nu)$ ▪

Exercise 2: Classify all $f:\tilde{\Lambda}\times\tilde{\Lambda}\to k^*$ (resp. $f:\Lambda\times\Lambda\to k^*$, $\Lambda:=q^{\mathbb{Z}}$) satisfying (1,2) for all weights $\lambda,\mu,\nu\in\Lambda^*$ (resp. $\lambda,\mu,\nu\in\Lambda$)

<u>Question</u>: What is the relation between two quantum groups we studied so far: $U_q(\mathfrak{sl}_2)$ and $SL_q(2)$?

As always, we should first understand its classical counterpart.

<u>Def</u>: (a) Given two bialgebras A, B, a bilinear form $\langle\cdot,\cdot\rangle : A\times B \to k$ is called a <u>bialgebra pairing</u> if the following equalities hold:

$$\boxed{\begin{array}{ll} \langle a, b_1 b_2\rangle = \sum\limits_{(a)} \langle a', b_1\rangle \cdot \langle a'', b_2\rangle , & \langle a_1 a_2, b\rangle = \sum\limits_{(b)} \langle a_1, b''\rangle \cdot \langle a_2, b'\rangle \\ \langle a, 1_B\rangle = \varepsilon(a) , & \langle 1_A, b\rangle = \varepsilon(b) \end{array}} \quad (1)$$

(b) If A, B – Hopf algebras, then $\langle\cdot,\cdot\rangle$ is a <u>Hopf pairing</u> if in addition

$$\boxed{\langle S(a), b\rangle = \langle a, S^{-1}(b)\rangle} \quad (2)$$

<u>Lemma 1</u>: (a) Given bialgebras A, B and a bilinear pairing $\langle\, ,\, \rangle : A\times B \to k$, it is a bialg. pairing iff the natural linear maps $\varphi : A^{op} \to B^*$ and $\psi : B \to A^*$ (induced by $\langle\, ,\, \rangle$) are algebra morphisms.

(b) If $\dim(B) < \infty$, then $\langle\, ,\, \rangle$ – bialgebra pairing iff $\varphi : A^{op} \to B^*$ is a bialgebra morphism.

▶ (a) Note that the second equality in (1) is equivalent to $\varphi(a_1 a_2) = \varphi(a_2)\varphi(a_1)$:

$\varphi(a_1 a_2)(b) = \langle a_1 a_2, b\rangle = \sum\limits_{(b)} \langle a_1, b''\rangle \cdot \langle a_2, b'\rangle = (\varphi(a_2)\varphi(a_1))(b)$

Also $\varphi(1_A)(b) = \langle 1_A, b\rangle = \varepsilon(b)$ implies that 4^{th} equality of (1) is equiv. to $\varphi(1_A) = 1_{B^{*}}$

Likewise, the first equality of (1) is equivalent to $\psi(b_1 b_2) = \psi(b_1)\psi(b_2)$:

$\psi(b_1 b_2)(a) = \langle a, b_1 b_2\rangle = \sum\limits_{(a)} \langle a', b_1\rangle\langle a'', b_2\rangle = (\psi(b_1)\psi(b_2))(a)$

Finally, the 3^{d} eq-n of (1) is equivalent to $\psi(1_B) = 1_{A^*}$, due to:

$\psi(1_B)(a) = \langle a, 1_B\rangle = \varepsilon(a)$

(b) It suffices to check that 1^{st} & 3^{d} conditions of (1) are equivalent to φ being a coalgebra morphism. The 1^{st} follows from:

$\Delta(\varphi(a))(b\otimes c) \overset{def}{=\!=} \varphi(a)(b\cdot c) = \langle a, bc\rangle = \sum\limits_{(a)} \langle a', b\rangle \cdot \langle a'', c\rangle = (\varphi\otimes\varphi)(\Delta(a))(b\otimes c)$

while the 3^{d} is equivalent to φ being compatible with counits ∎

Consider the homomorphism $\boxed{\varphi: M(2)^{op} \to U(sl_2)^*}$ determined by $\varphi: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = \rho(u)$, $\rho: U(sl_2) \to \mathrm{End}(\underset{k^2}{V_1})$, V_1 - irr. 2-dim sl_2-repr.

Note: (1) $M(2)$ - commutative $\Rightarrow M(2)^{op} \simeq M(2)$
(2) $U(sl_2)$ - cocommutative $\Rightarrow U(sl_2)^*$ - commutative $\Big\} \Rightarrow \varphi$ is well-defined.

Prop 3: Define the pairing $\langle , \rangle : M(2) \times U(sl_2) \to k$ via $\langle h, u \rangle := \varphi(h)(u)$.
This is a bialgebra pairing.

▶ According to the proof of Lemma 1, 2^{nd} and 4^{th} conditions of (1) hold.

- The 3^{d} condition of (1) follows from the fact $\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \langle a,1\rangle & \langle b,1\rangle \\ \langle c,1\rangle & \langle d,1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the fact that $\varepsilon(\cdot), \langle \cdot, 1\rangle$ - homom, where the latter follows from 2^{d} condition of (1): $\langle a_1 a_2, 1\rangle = \langle a_1, 1\rangle \cdot \langle a_2, 1\rangle$.
- To check the 1^{st} condition of (1), note that

$$\left. \begin{array}{l} \rho(uv) = \rho(u)\rho(v) \Rightarrow \begin{pmatrix} \langle a, uv\rangle & \langle b, uv\rangle \\ \langle c, uv\rangle & \langle d, uv\rangle \end{pmatrix} = \begin{pmatrix} \langle a, u\rangle & \langle b, u\rangle \\ \langle c, u\rangle & \langle d, u\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle a, v\rangle & \langle b, v\rangle \\ \langle c, v\rangle & \langle d, v\rangle \end{pmatrix} \\ \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \right\} \Rightarrow$$

$\Rightarrow 1^{st}$ condition of (1) holds for a being one of the 4 generators $\{a,b,c,d\}$.
The result now follows from the following claim:

CLAIM: If $\langle x, b_1 b_2\rangle = \sum_{(x)} \langle x', b_1\rangle \langle x'', b_2\rangle$, $\langle y, b_1 b_2\rangle = \sum_{(y)} \langle y', b_1\rangle \langle y'', b_2\rangle$ $\forall b_1, b_2$,
then $\langle xy, b_1 b_2\rangle = \sum_{(xy)} \langle (xy)', b_1\rangle \langle (xy)'', b_2\rangle$.

Indeed,

$$\langle xy, b_1 b_2\rangle = \sum_{(b_1 b_2)} \langle x, (b_1 b_2)''\rangle \cdot \langle y, (b_1 b_2)'\rangle = \sum_{(b_1)(b_2)} \langle x, b_1'' \cdot b_2''\rangle \cdot \langle y, b_1' \cdot b_2'\rangle$$

$$= \sum_{(b_1)(b_2)(x)(y)} \langle x', b_1''\rangle \cdot \langle x'', b_2''\rangle \cdot \langle y', b_1'\rangle \cdot \langle y'', b_2'\rangle$$

$$= \sum_{(x)(y)} \langle x'y', b_1\rangle \cdot \langle x''y'', b_2\rangle = \sum_{(xy)} \langle (xy)', b_1\rangle \cdot \langle (xy)'', b_2\rangle \qquad \checkmark$$

This completes the proof of Proposition ∎

Lemma 2: $\varphi(ad - bc) = 1$.

▶ Need to verify $\langle ad - bc, u\rangle = \varepsilon(u)$ $\forall u \in U(sl_2)$. Since $ad - bc$ is a group-like el-t it suffices to check the latter equality for $u = 1, e, f, h$. This is a straightforward computation. E.g: $\langle ad - bc, h\rangle = \langle a \otimes d - b \otimes c, h \otimes 1 + 1 \otimes h\rangle =$
$= 1 \cdot \varepsilon(d) + \varepsilon(a) \cdot (-1) - 0 \cdot \varepsilon(c) - \varepsilon(b) \cdot 0 = 1 - 1 = 0$ ∎

Due to Lemma 2, we see that φ factors through

$$\boxed{\varphi: SL(2)^{op} = M(2)^{op}/(ad-bc-1) \longrightarrow U(sl_2)^*.}$$

Prop 4: Define the pairing $\langle , \rangle : SL(2) \times U(sl_2) \to k$ via $\langle h, u\rangle := \varphi(h)(u)$.

This is a Hopf pairing.

▶ The fact that $\langle , \rangle$ is a bialgebra pairing follows immediately from Prop 3. Indeed, due to Lemma 1, we need to verify that $\psi: U(sl_2) \to SL(2)^*$ is an algebra homomorphism. But we know that composing it further with injective morphism $SL(2)^* \hookrightarrow M(2)^*$ is an alg. homom (Prop 3), hence, the result.

It remains to verify the compatibility with antipodes, condition (2).

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, S^{-1}(h) \right\rangle = \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, -h \right\rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left\langle S\begin{pmatrix} a & b \\ c & d \end{pmatrix}, h \right\rangle = \left\langle \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, h \right\rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

One similarly checks $\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, S^{-1}(e)\right\rangle = \left\langle S\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e\right\rangle$, $\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, S^{-1}(f)\right\rangle = \left\langle S\begin{pmatrix} a & b \\ c & d \end{pmatrix}, f\right\rangle$.

Knowing it on generators, we can prove in general, due to:

- $\langle S(xy), u\rangle = \langle S(y)S(x), u\rangle = \sum_{(u)} \langle S(y), u''\rangle \cdot \langle S(x), u'\rangle = \sum_{(u)} \langle y, S^{-1}(u'')\rangle \langle x, S^{-1}(u')\rangle$
  $= \langle xy, S^{-1}(u)\rangle$ due to the fact $\Delta \circ S = (S \otimes S)\Delta^{op}$.
- $\langle S(x), uv\rangle = \sum_{(x)} \langle S(x''), u\rangle \cdot \langle S(x'), v\rangle = \sum_{(x)} \langle x'', S^{-1}(u)\rangle \cdot \langle x', S^{-1}(v)\rangle$
  $= \langle x, S^{-1}(v) \cdot S^{-1}(u)\rangle = \langle u, S^{-1}(uv)\rangle$ due to $S\mu = \mu^{op}(S \otimes S)$. ∎

We have established a Hopf pairing b/w $SL(2)$ and $U_q(sl_2)$.
In particular, this yields algebra morphisms $SL(2)^{op} \to U(sl_2)$ and $U(sl_2) \to SL(2)^*$.

In particular, using the latter algebra morphism, any $SL(2)^*$-modules becomes an $U(sl_2)$-module. Recall the comodule $k[x,y]_n$ of $SL(2)$ $\Rightarrow$ $\Rightarrow$ its dual $k[x,y]_n^*$ is an $SL(2)^*$-module $\Rightarrow U(sl_2) \curvearrowright k[x,y]_n^*$.

Exercise 3: Verify that $k[x,y]_n^* \simeq V_n$ as sl_2-modules

Rmk: An invariant way to think about the introduced pairing b/w $SL(2)$ and $U(sl_2)$ is by realizing any el-t $u \in U(sl_2)$ as a left/right-invariant differential operator D_u on Spec$(SL(2))$ and evaluating $(D_u(h))$ at 1 $\forall$ $h \in SL(2)$ - function on Spec $(SL(2))$

It turns out that all the results from pp. 4–6 generalize to the q-case in a straightforward way (though the proofs are more technical).

Exercise 4: (a) Prove that there is an algebra automorphism $\varphi: M_q(2)^{op} \longrightarrow U_q(sl_2)^*$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $\forall u \in U_q(sl_2)$ we define $\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = \rho(u)$ with $\rho: U_q(sl_2) \longrightarrow \mathrm{End}(L(1,+))$.

(b) Verify that the induced pairing $\langle \circ, \circ \rangle : M_q(2) \times U_q(sl_2) \longrightarrow k$ is a bialgebra pairing

(c) Verify $\varphi(\det_q) = 1$

(d) Verify that the induced pairing $\langle \circ, \circ \rangle : SL_q(2) \times U_q(sl_2) \longrightarrow k$ is a Hopf pairing.

(e) Let us consider an $SL_q(2)$-comodule $k_q[x,y]_n$, dualizing which, we get $SL_q(2)^* \curvearrowright k_q[x,y]_n^*$. On the other hand, pairing of (d) gives rise to an alg. homom $\psi: U_q(sl_2) \longrightarrow SL_q(2)^*$. Prove that the induced action of $U_q(sl_2)$ on $k_q[x,y]_n^*$ is isomorphic to $L(n,+)$.