# — LECTURE 10 — (02/21/2018)

Recall: Simple $\mathbb{C}$-Lie algebras (f.d.) are determined by their root systems

$\Delta \supset \Pi = \{\alpha_1, \dots, \alpha_\ell\}$ — positive simple roots

root system

Chevalley generators of $\mathfrak{g}$: $\{h_\alpha, e_\alpha, f_\alpha\}_{\alpha\in\Pi}$ subject to the following defining rels:

$$\boxed{\begin{array}{l}[h_\alpha, h_\beta]=0, \quad [h_\alpha, e_\beta]=a_{\alpha\beta}\cdot e_\beta, \quad [h_\alpha, f_\beta] = -a_{\alpha\beta} f_\beta, \quad [e_\alpha, f_\beta] = \delta_{\alpha\beta}\cdot h_\alpha \\ (\mathrm{ad}\, e_\alpha)^{1-a_{\alpha\beta}} e_\beta = 0 = (\mathrm{ad}\, f_\alpha)^{1-a_{\alpha\beta}} f_\beta \quad (\text{for } \alpha\neq\beta)\end{array}}$$

Here $a_{\alpha\beta}$ are the entries of the corresponding Cartan matrix, which are expressed through the inner form $(\cdot,\cdot)$ on $\mathbb{R}\Delta$ via $\boxed{a_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}}$ We recall that such form is uniquely determined by two conditions:

- invariant w.r.t. action of the Weyl gp $W$
- $(\alpha,\alpha)=2$ for short roots $\alpha\in\Pi$.

It is known that $\forall \beta\in\Pi$, the value $(\beta,\beta)$ is one of $\{2,4,6\}$. Hence, the number $d_\beta := \frac{(\beta,\beta)}{2}$ is among $\{1,2,3\}$.

We will also use the standard notations for root/weight lattices $P\supset Q$:

$Q :=$ root lattice $= \mathbb{Z}$-lattice with basis $\{\alpha\}_{\alpha\in\Pi} = \bigoplus_{\alpha\in\Pi}\mathbb{Z}\alpha$

$P :=$ weight lattice $= \mathbb{Z}$ - lattice with basis $\{\omega_\alpha\}_{\alpha\in\Pi} = \bigoplus_{\alpha\in\Pi}\mathbb{Z}\omega_\alpha$

fundamental weights, s.t. $(\omega_\alpha,\beta)=\delta_{\alpha\beta}\cdot d_\beta$.

We will assume that $q^{2d_\alpha}\neq 1$ $\forall\alpha\in\Pi$ and set $\boxed{q_\alpha := q^{d_\alpha} = q^{\frac{(\alpha,\alpha)}{2}}, \quad [a]_\alpha := [a]_{q_\alpha}, \text{etc.}}$

**Def:** The quantized enveloping algebra $U_q(\mathfrak{g})$ (of Drinfeld & Jimbo) is defined as the $k$-algebra generated by $\{E_\alpha, F_\alpha, K_\alpha^{\pm1}\}_{\alpha\in\Pi}$ subject to:

$$K_\alpha K_\alpha^{-1} = K_\alpha^{-1}K_\alpha = 1, \quad K_\alpha K_\beta = K_\beta K_\alpha$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta, \quad K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta$$

$$[E_\alpha, F_\beta] = \delta_{\alpha\beta}\cdot \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

$$\left.\begin{array}{l}\sum_{r=0}^{1-a_{\alpha\beta}} (-1)^r \begin{bmatrix}1-a_{\alpha\beta}\\ r\end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-r} E_\beta E_\alpha^r = 0 \\ \sum_{r=0}^{1-a_{\alpha\beta}} (-1)^r \begin{bmatrix}1-a_{\alpha\beta}\\ r\end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-r} F_\beta F_\alpha^r = 0\end{array}\right\} q\text{-Serre rels}$$

**Def:** Let $\overline{U}_q(\mathfrak{g})$ be an alg. defined in the same way, but without $q$-Serre rels

Note that we have a natural projection $\overline{U}_q(\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g})$

In what follows, we will often prove the results for $\overline{U}_q(\mathfrak{g})$ first, and then deduce for $U_q(\mathfrak{g})$.

- Let $U_q^-$, $U_q^0$, $U_q^+$ be the subalg-s of $U_q(\mathfrak{g})$, generated by $\{F_\alpha\}_{\alpha\in\Pi}$, $\{K_\alpha^{\pm1}\}_{\alpha\in\Pi}$, $\{E_\alpha\}_{\alpha\in\Pi}$, respectively. Similarly, define the subalg-s $\overline{U}_q^-$, $\overline{U}_q^0$, $\overline{U}_q^+$ of $\overline{U}_q(\mathfrak{g})$.
- For any $\lambda=\sum_{\alpha\in\Pi} n_\alpha\cdot\alpha\in Q$ $(n_\alpha\in\mathbb{Z})$, set $K_\lambda:=\prod_{\alpha\in\Pi} K_\alpha^{n_\alpha}$ – viewed as el-t of $\overline{U}_q^0$ or $U_q^0$.

<u>Lemma 1</u>: For every $\alpha\in\Pi$, there is a natural homomorphism $\boxed{\iota_\alpha: U_{q_\alpha}(\mathfrak{sl}_2)\longrightarrow U_q(\mathfrak{g})}$ (and also $U_{q_\alpha}(\mathfrak{sl}_2)\to\overline{U}_q(\mathfrak{g})$) determined by $\boxed{E\mapsto E_\alpha,\ F\mapsto F_\alpha,\ K^{\pm1}\longmapsto K_\alpha^{\pm1}}$

▸ Obvious! ∎

<u>Cor</u>: We have $[E_\alpha, F_\alpha^r]=[r]_\alpha\cdot F_\alpha^{r-1}\cdot \underbrace{[K_\alpha;1-r]}_{\frac{K_\alpha q_\alpha^{1-r}-K_\alpha^{-1}q_\alpha^{r-1}}{q_\alpha-q_\alpha^{-1}}}$, $[F_\alpha, E_\alpha^r]=-[r]_\alpha E_\alpha^{r-1}\cdot[K_\alpha;r-1]$
both in $U_q(\mathfrak{g})$ and $\overline{U}_q(\mathfrak{g})$

<u>Lemma 2</u>: (a) Both $U_q(\mathfrak{g})$ and $\overline{U}_q(\mathfrak{g})$ admit unique automorphisms (Cartan involut.) determined by $\boxed{\omega: E_\alpha\longmapsto F_\alpha,\ F_\alpha\mapsto E_\alpha,\ K_\alpha^{\pm1}\longmapsto K_\alpha^{\mp1}}$.

(b) Both $U_q(\mathfrak{g})$ and $\overline{U}_q(\mathfrak{g})$ admit unique antiautomorphisms determined by $\boxed{\sigma: E_\alpha\mapsto E_\alpha,\ F_\alpha\mapsto F_\alpha,\ K_\alpha\mapsto K_\alpha^{\pm1}}$.

▸ Straightforward! (verify at home!) ∎

<u>Rmks</u>: (a) The algebras $U_q(\mathfrak{g})$ and $\overline{U}_q(\mathfrak{g})$ are naturally $Q$-graded with $\boxed{\deg(E_\alpha)=\alpha,\ \deg(F_\alpha)=-\alpha,\ \deg(K_\alpha^{\pm1})=0.}$ Note that, we have $\boxed{K_\lambda x K_\lambda^{-1}=q^{(\lambda,\mu)}\cdot x \quad \text{if } \deg(x)=\mu}$

(b) Sometimes, people consider a slightly extended version of $U_q(\mathfrak{g})$ with a "bigger" Cartan part. Namely, for every $\Gamma$ (subgp of $P$): $\boxed{P\supseteq\Gamma\supseteq Q}$. For each such $\Gamma$, the algebra $\boxed{U_q^\Gamma(\mathfrak{g})}$ is generated by $\{E_\alpha, F_\alpha, K_\beta^{\pm1}\}_{\alpha\in\Pi}^{\beta\in\Gamma}$, where the new rel-s read $K_{\beta_1}K_{\beta_2}=K_{\beta_2}K_{\beta_1}$, $K_\beta E_\alpha K_\beta^{-1}=q^{(\beta,\alpha)}E_\alpha$, $K_\beta F_\alpha K_\beta^{-1}=q^{-(\beta,\alpha)}F_\alpha$. The standard choice is $\Gamma=Q$ (when it is skipped in notations)

Goal 1: Want to endow $U_q(\mathfrak{g})$ with a Hopf alg. structure.

We will first provide a Hopf alg. str. on $\overline{U}_q(\mathfrak{g})$. It is natural to expect that the natural algebra homomorphisms $U_{q_\alpha}(\mathfrak{sl}_2) \xrightarrow{\imath_\alpha} \overline{U}_q(\mathfrak{g})$ are Hopf alg. morphisms. The latter observation yields explicit f-las for $\Delta, S, \varepsilon$.

Thm 1: There is a unique Hopf alg. structure on $\overline{U}_q(\mathfrak{g})$ with the coproduct $\Delta$, antipode $S$, counit $\varepsilon$ determined by:

$\Delta$: $E_\alpha \mapsto E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha$, $F_\alpha \mapsto F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha$, $K_\alpha^{\pm 1} \mapsto K_\alpha^{\pm 1} \otimes K_\alpha^{\pm 1}$

$S$: $E_\alpha \mapsto -K_\alpha^{-1} E_\alpha$, $F_\alpha \mapsto -F_\alpha K_\alpha$, $K_\alpha^{\pm 1} \mapsto K_\alpha^{\mp 1}$

$\varepsilon$: $E_\alpha \mapsto 0$, $F_\alpha \mapsto 0$, $K_\alpha^{\pm 1} \mapsto 1$

► We need to prove that each of these assignments is compatible with the defining relations of $U_q(\mathfrak{g})$. Note that compatibility with rel-s involving two equal roots $\alpha = \beta$ follows from the fact that we have alg. homom. $\imath_\alpha : U_{q_\alpha}(\mathfrak{sl}_2) \to U_q(\mathfrak{g})$ and we already established Hopf alg. str. on $U_{q_\alpha}(\mathfrak{sl}_2)$.

The only non-trivial relation for $\alpha \neq \beta$ is $[E_\alpha, F_\beta] = 0$ $(\beta \neq \alpha)$.

- For $\Delta$: $[\Delta(E_\alpha), \Delta(F_\beta)] = [E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, F_\beta \otimes K_\beta^{-1} + 1 \otimes F_\beta] =$

$$= \underbrace{[E_\alpha, F_\beta]}_{0} \otimes K_\beta^{-1} + K_\alpha \otimes \underbrace{[E_\alpha, F_\beta]}_{0} + K_\alpha F_\beta \otimes E_\alpha K_\beta^{-1} - F_\beta K_\alpha \otimes K_\beta^{-1} E_\alpha =$$

$$= q^{-(\alpha,\beta)} F_\beta K_\alpha \otimes E_\alpha K_\beta^{-1} - F_\beta K_\alpha \otimes q^{-(\alpha,\beta)} E_\alpha K_\beta^{-1} = 0. \quad \checkmark$$

- For $S$: $[S(E_\alpha), S(F_\beta)] = [-K_\alpha^{-1} E_\alpha, -F_\beta K_\beta]_{\mathrm{opp.alg}} = F_\beta K_\beta K_\alpha^{-1} E_\alpha - K_\alpha^{-1} E_\alpha F_\beta K_\beta =$

$$= q^{-(\alpha,\alpha)+(\beta,\alpha)} F_\beta E_\alpha K_\beta K_\alpha^{-1} - q^{-(\alpha,\alpha)+(\beta,\alpha)} E_\alpha F_\beta K_\alpha^{-1} K_\beta = 0 \quad (\text{as } [E_\alpha, F_\beta] = 0)$$

- For $\varepsilon$: Obvious

∎

Cor: Due to $\imath_\alpha$ being a Hopf alg. homomorphism we immediately obtain the formulas for $\Delta(E_\alpha^{(r)})$, $\Delta(F_\alpha^{(r)})$, $S(E_\alpha^{(r)})$, $S(F_\alpha^{(r)})$ ← read them from those we had for $U_{q_\alpha}(\mathfrak{sl}_2)$.

Lemma 3: $S^2(x) = K_{2\rho}^{-1} x K_{2\rho}$ $\forall x \in \overline{U}_q(\mathfrak{g})$, where $\rho = \sum_{\alpha \in \Pi} \omega_\alpha = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$

► $S^2(K_\alpha) = S(K_\alpha^{-1}) = K_\alpha$

$S^2(E_\alpha) = S(-K_\alpha^{-1} E_\alpha) = K_\alpha^{-1} E_\alpha K_\alpha = q^{-(\alpha,\alpha)} E_\alpha = q^{-(2\rho,\alpha)} E_\alpha = K_{2\rho}^{-1} E_\alpha K_{2\rho}$ as $(2\rho, \alpha) = (\alpha, \alpha)$

$S^2(F_\alpha) = S(-F_\alpha K_\alpha) = K_\alpha^{-1} F_\alpha K_\alpha = q^{(\alpha,\alpha)} F_\alpha = q^{(2\rho,\alpha)} F_\alpha = K_{2\rho}^{-1} F_\alpha K_{2\rho}$

∎

Note that $U_q(\mathfrak{g}) \simeq \overline{U}_q(\mathfrak{g})/I$, where $I$ is the 2-sided ideal gen-d by $\{u^{\pm}_{\alpha\beta}\}_{\alpha\neq\beta\in\Pi}$:

$$\boxed{\begin{aligned} u^{+}_{\alpha\beta} &:= \sum_{r=0}^{1-a_{\alpha\beta}} (-1)^r \begin{bmatrix} 1-a_{\alpha\beta} \\ r \end{bmatrix}_{\alpha} E_\alpha^{1-a_{\alpha\beta}-r} E_\beta E_\alpha^r \in \overline{U}_q(\mathfrak{g}) \\ u^{-}_{\alpha\beta} &:= \sum_{r=0}^{1-a_{\alpha\beta}} (-1)^r \begin{bmatrix} 1-a_{\alpha\beta} \\ r \end{bmatrix}_{\alpha} F_\alpha^{1-a_{\alpha\beta}-r} F_\beta F_\alpha^r \in \end{aligned}}$$

<u>Lemma 4</u>: For $\alpha\neq\beta\in\Pi$, we have:

$$\Delta(u^{+}_{\alpha\beta}) = u^{+}_{\alpha\beta}\otimes 1 + K_\alpha^{1-a_{\alpha\beta}} K_\beta \otimes u^{+}_{\alpha\beta}, \quad \Delta(u^{-}_{\alpha\beta}) = u^{-}_{\alpha\beta}\otimes K_\alpha^{-1+a_{\alpha\beta}} K_\beta^{-1} + 1\otimes u^{-}_{\alpha\beta}$$

<u>Exercise 1</u>: Prove Lemma 4.

<u>Lemma 5</u>: For $\alpha\neq\beta\in\Pi$, we have: $S(u^{+}_{\alpha\beta}) = -K_\alpha^{-1+a_{\alpha\beta}} K_\beta^{-1} u^{+}_{\alpha\beta}$
$S(u^{-}_{\alpha\beta}) = -u^{-}_{\alpha\beta} K_\alpha^{1-a_{\alpha\beta}} K_\beta$

▶ Apply f-la $\sum_{(x)} S(x')x'' = \eta\varepsilon(x)$ for $x = u^{+}_{\alpha\beta}$ together with f-la for $\Delta(u^{+}_{\alpha\beta})$ from Lemma 4, to get (note $\varepsilon(u^{+}_{\alpha\beta})=0$):

$$0 = S(u^{+}_{\alpha\beta})\cdot 1 + S(K_\alpha^{1-a_{\alpha\beta}} K_\beta)\cdot u^{+}_{\alpha\beta} = S(u^{+}_{\alpha\beta}) + K_\alpha^{-1+a_{\alpha\beta}} K_\beta^{-1}\cdot u^{+}_{\alpha\beta} \Rightarrow S(u^{+}_{\alpha\beta}) = -K_\alpha^{-1+a_{\alpha\beta}} K_\beta^{-1} u^{+}_{\alpha\beta}.$$

To get f-la for $S(u^{-}_{\alpha\beta})$ apply $\sum_{(x)} x' S(x'') = \eta\varepsilon(x)$ for $x = u^{-}_{\alpha\beta}$ (& Lemma 4) ∎

According to Lemmas 4, 5, and observation that $\varepsilon(u^{\pm}_{\alpha\beta})=0$, we have:
$\varepsilon(I)=0$, $S(I)\subset I$, $\Delta(I)\subset I\otimes \overline{U}_q(\mathfrak{g}) + \overline{U}_q(\mathfrak{g})\otimes I$.

Hence, the Hopf alg. structure on $\overline{U}_q(\mathfrak{g})$ of Thm 1 descends to the one on $\overline{U}_q(\mathfrak{g})/I \simeq U_q(\mathfrak{g})$.

<u>Thm 2</u>: There is a unique Hopf alg. structure on $U_q(\mathfrak{g})$ determined on the generators by the same formulas as in Thm 1.

<u>Goal 2</u>: Establish the triangular decomposition for $U_q(\mathfrak{g})$.

Today, we will establish the one for $\overline{U}_q(\mathfrak{g})$, while in the beginning of next class we will deduce from it the statement for $U_q(\mathfrak{g})$

The proof for $\overline{U}_q(\mathfrak{g})$ will be based on the next few lemmas.

For a sequence $I=(\beta_1,\dots,\beta_r)\in\Pi^r$ $(r\geq 0)$, set $E_I:=E_{\beta_1}\cdots E_{\beta_r}$ $(E_\emptyset=1)$
$F_I:=F_{\beta_1}\dots F_{\beta_r}$ $(F_\emptyset=1)$
$wt(I):=\beta_1+\dots+\beta_r\in Q.$

Lemma 6: $\forall I\ \exists\ C^I_{A,B}(v)\in\mathbb{Z}[v,v^{-1}]$ whose coeff-s do not depend on $q$ and which are parametrized by finite sequences $A,B$ as above with $wt(A)+wt(B)=wt(I)$ and such that both in $\overline{U}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ we have:

$$\boxed{\Delta(E_I)=\sum_{A,B}C^I_{A,B}(q)\,E_AK_{wt(B)}\otimes E_B,\quad \Delta(F_I)=\sum_{A,B}C^I_{A,B}(q^{-1})\,F_A\otimes K^{-1}_{wtA}F_B}$$

Also $C^I_{A,\emptyset}=\delta_{A,I}$, $C^I_{\emptyset,B}=\delta_{B,I}$.

▷ Proved by induction on $r$. For $r=1$ - follows from our f-las for $\Delta(E_\alpha)$, $\Delta(F_\alpha)$.
To make an induction step, write $I=(\beta_1,\dots,\beta_r)$, $J=(\beta_2,\dots,\beta_r)$, so that
$\Delta(E_I)=\Delta(E_{\beta_1})\Delta(E_J)$, $\Delta(F_I)=\Delta(F_{\beta_1})\Delta(F_J)$.

! Complete the verification at home! ◻

Cor: For any $\lambda\in Q_+$ $(\lambda\in Q,\lambda\geq 0)$, we have

$$\boxed{\Delta(U^+_q(\mathfrak{g})_\lambda)\subset\bigoplus_{0\leq\mu\leq\lambda}U^+_q(\mathfrak{g})_{\lambda-\mu}\cdot K_\mu\otimes U^+_q(\mathfrak{g})_\mu,\quad \Delta(U^-_q(\mathfrak{g})_{-\lambda})\subset\bigoplus_{0\leq\mu\leq\lambda}U^-_q(\mathfrak{g})_{-\mu}\otimes U^-_{-(\lambda-\mu)}K^{-1}_\mu}$$

(Recall: $\lambda\geq\mu\iff\lambda-\mu\geq 0\iff\lambda-\mu=\sum_{\alpha\in\Pi}n_\alpha\cdot\alpha$ with $n_\alpha\in\mathbb{Z}_{\geq 0}\ \forall\alpha\in\Pi$).

Lemma 7 (Verma modules for $\overline{U}_q(\mathfrak{g})$): Choose a collection $\{c_\alpha\}_{\alpha\in\Pi}\in(k^*)^\Pi$ and let $M_c$ be the vector space with basis $(v_I)_{I\text{-fin. sequences of simple roots, includ. }\emptyset}$. Define the following operators on $M_c$:

$$\boxed{F_\alpha(v_I)=v_{(\alpha,I)},\quad K^{\pm1}_\alpha(v_I)=c^{\pm1}_\alpha q^{\mp(\alpha,wtI)}\cdot v_I,\quad E_\alpha(v_I)=\sum_{j:\beta_j=\alpha}\frac{c_\alpha q^{-(\alpha,\mu_j)}-c^{-1}_\alpha q^{(\alpha,\mu_j)}}{q_\alpha-q^{-1}_\alpha}v_{(\beta_1,\dots,\hat{\beta}_j,\dots,\beta_r)}}$$

where $I=(\beta_1,\dots,\beta_r)$, $\mu_j=\beta_{j+1}+\dots+\beta_r$. Then: these operators define action $\overline{U}_q(\mathfrak{g})\curvearrowright M_c$.

▷ It is clear that $\{K_\alpha\}_{\alpha\in\Pi}$ pairwise commute, as well as satisfy
$K_\alpha E_\beta K^{-1}_\alpha=q^{(\alpha,\beta)}E_\beta$, $K_\alpha F_\beta K^{-1}_\alpha=q^{-(\alpha,\beta)}F_\beta$.

It remains to check $E_\alpha F_\beta-F_\beta E_\alpha=\delta_{\alpha\beta}\cdot\frac{K_\alpha-K^{-1}_\alpha}{q_\alpha-q^{-1}_\alpha}$ (*)

- If $\alpha\neq\beta$, then the summands in $E_\alpha(F_\beta(v_I))$ exactly match those of $F_\beta(E_\alpha(v_I))$ $\forall I$
- If $\alpha=\beta$, then $E_\alpha(F_\beta(v_I))$ has one more new term in comparison to $F_\beta(E_\alpha(v_I))$, where applying $E_\alpha$ we erase $\alpha$ "added by action of $F_\alpha$". This matches (*) ◻

Twisting this representation by a Cartan involution (and renaming $c_\alpha^{-1} \rightsquigarrow c_\alpha$), get:

Lemma 8 (Dual Verma): Choose a collection $\{c_\alpha\}_{\alpha\in\Pi} \in (k^*)^\Pi$ and let $M'_c$ be a vector space with the same basis as $M_c$. Define the following operators on $M'_c$:

$$\boxed{E_\alpha(v_I) = v_{(\alpha,I)},\quad K_\alpha^{\pm1}(v_I) = c_\alpha^{\pm1} q^{\pm(\alpha,\mathrm{wt} I)} v_I,\quad F_\alpha(v_I) = \sum_{j:\beta_j=\alpha} \frac{c_\alpha^{-1} q^{-(\alpha,\mu_j)} - c_\alpha q^{(\alpha,\mu_j)}}{q_\alpha - q_\alpha^{-1}} v_{(\beta_1,\ldots,\hat{\beta}_j,\ldots,\beta_r)}}$$

Then: These operators define the action $\bar{U}_q(\mathfrak{g}) \curvearrowright M'_c$.

Finally, we are ready to prove the following result:

Thm 3: The elements $\{F_I K_\mu E_J \mid I,J$ - finite sequences of simple roots, $\mu\in Q\}$ are a basis of $\bar{U}_q(\mathfrak{g})$.

Corollary (Triangular decomposition of $\bar{U}_q(\mathfrak{g})$):

(a) $\bar{U}_q^-$ is a free algebra gen-d by $F_\alpha$: $\bar{U}_q^- \cong k\langle F_\alpha\rangle_{\alpha\in\Pi}$

(b) $\bar{U}_q^+$ is isomorphic to a free alg. gen-d by $E_\alpha$: $\bar{U}_q^+ \simeq k\langle E_\alpha\rangle_{\alpha\in\Pi}$

(c) $\bar{U}_q^0$ is isomorphic to Laurent polynomials in $K_\alpha$: $\bar{U}_q^0 \simeq k\langle K_\alpha^{\pm1}\rangle_{\alpha\in\Pi}$.

(d) The multiplication map $\bar{U}_q^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ \longrightarrow \bar{U}_q(\mathfrak{g})$ is an isom. of v.spaces.

Cor: The homom-s $\iota_\alpha : U_{q_\alpha}(\mathfrak{sl}_2) \longrightarrow \bar{U}_q(\mathfrak{g})$ are injective.

▶ (Proof of Thm 3).

- First, we check that $V := \mathrm{span}\langle F_I K_\mu E_J\rangle$ coincides with $\bar{U}_q(\mathfrak{g})$. This follows from the fact that $1\in V$ and $V$ is invariant w.r.t. left multiplication of $\bar{U}_q(\mathfrak{g})$. The latter immediately follows from the defining rel-s
- To prove $\{F_I K_\mu E_J\}$ are lin. ind, assume the contrary: there exists a nontrivial linear combination $X = \sum_{I,J,\mu} a_{I,J,\mu} F_I K_\mu E_J$ equal to ZERO.

Choose $I_0$, s.t. $\exists J,\mu : a_{I_0,J,\mu} \neq 0$ and $\mathrm{wt}(I_0)$ - maximal.

Choose a collection $\{c_\alpha\}_{\alpha\in\Pi} \in (k^*)^\Pi$ and $\{c'_\alpha := 1\}$ $\forall \alpha\in\Pi$. Then, due to Lemmas 7,8 and Hopf alg. str. on $\bar{U}_q(\mathfrak{g})$, we have the action

$$\boxed{\bar{U}_q(\mathfrak{g}) \curvearrowright M_{c'} \otimes M'_c.}$$

Let us compute $X(v_\emptyset \otimes v'_\emptyset)$. On one hand it is ZERO as $X=0$.

But on the other hand:

$$X(v_\emptyset \otimes v'_\emptyset) = \sum_{I,J,\mu} a_{I,J,\mu} F_I K_\mu E_J (v_\emptyset \otimes v'_\emptyset) \xrightarrow[E_\alpha(v_\emptyset)=0\,\forall\alpha]{\text{Lemma 6}} \sum_{I,J,\mu} a_{I,J,\mu} F_I K_\mu (v_\emptyset \otimes v_J)$$

$$= \sum_{I,J,\mu} a_{I,J,\mu} \cdot F_I(v_\emptyset \otimes c^\mu \cdot q^{(\mu, wtJ)} v_J) \quad \left(\begin{array}{l}\text{where for } \mu = \sum_{\alpha\in\Pi} n_\alpha \cdot \alpha, \\ \text{we set } c^\mu := \prod_\alpha c_\alpha^{n_\alpha}\end{array}\right)$$

Applying again Lemma 6 and considering only terms with the first factor in $M_{c'}$ being $v_\emptyset$ (or its multiple), we get (revoking $wt(I_0)$-max.):

$$0 = \sum_{J,\mu} a_{I_0,J,\mu} \cdot c^\mu \cdot q^{(\mu, wtJ)} \cdot v_{I_0} \otimes K^{-1}_{wt(I_0)} v_J$$

$$\Downarrow$$

$$\boxed{\forall J_0: \quad 0 = \sum_\mu a_{I_0,J_0,\mu} \cdot c^\mu \cdot q^{(\mu, wtJ_0)} \cdot q^{-(wtI_0, wtJ_0)} = 0} \quad (\dagger)$$

The right-hand side is a Laurent polynomial in $\{c_\alpha\}_{\alpha\in\Pi}$.

Thus, if $|k| = \infty$, then $(\dagger)$ for any collection $\{c_\alpha\} \in (k^*)^\Pi$ implies that all coefficients $a_{I_0,J_0,\mu}$ are ZERO, contradicting our assumption on $I_0$. If $|k| < \infty$, we can apply the above argument to representations $M_c \otimes_k \bar{k}$, $M'_c \otimes_k \bar{k}$ for any field extension $\bar{k} \supset k$, reducing to the previous case. $\blacksquare$