— LECTURE 11 — (02/26/2018)

Last time: Proved the triangular decomposition for the algebre $\overline{U}_q(\mathfrak{g})$, i.e.

$$\text{mult}: \underset{\text{free alg. in } \{F_\alpha\}_{\alpha\in\Pi}}{\overline{U}_q^-} \otimes \underset{\text{Laurent pol's in } \{K_\alpha \mid \alpha\in\Pi\}}{\overline{U}_q^0} \otimes \underset{\text{free alg. in } \{E_\alpha\mid\alpha\in\Pi\}}{\overline{U}_q^+} \longrightarrow \overline{U}_q(\mathfrak{g}) \text{ – isom. of vector spaces}$$

Want: Deduce a similar statement for the algebre $U_q(\mathfrak{g})$.

We will use the notion of adjoint representation of a Hopf algebra (H,Δ,ε,S).

Def: The adjoint action of H on itself is given by $\boxed{ad(x)(y)=\sum_{(x)} x'yS(x'')}$

One can think about this as an action of $H\otimes H \curvearrowright H$ via $(x_1\otimes x_2)(y)=x_1yS(x_2)$, composed further with alg. homom. $\Delta: H\to H\otimes H$

Rmks: (a) In case $H=\mathbb{C}[G]$, we recover $ad(g)(y)=gyg^{-1}$ $(g\in G)$

(b) In case $H=U\mathfrak{g}$, we recover $ad(x)(y)=xy-yx$ $(x\in\mathfrak{g})$

(c) Recall the H-action on $Hom(V,V')$ for two H-modules V,V' from Lecture 2.
Explicitly for $f\in Hom(V,V')$, $(xf)(v)=\sum_{(x)} x' f(S(x'')v)$.
Therefore, given H-module M, the map $H\to End(M)$ is an H-module homomorphism

Let us now get some formulas for the adjoint representation of $\overline{U}_q(\mathfrak{g})$.

Lemma 1: For any $\alpha\in\Pi$ and $x\in\overline{U}_q(\mathfrak{g})$ or $U_q(\mathfrak{g})$, we have:

$$\boxed{ad(E_\alpha)x=E_\alpha x-K_\alpha xK_\alpha^{-1}E_\alpha,\quad ad(F_\alpha)x=(F_\alpha x-xF_\alpha)K_\alpha,\quad ad(K_\alpha)x=K_\alpha xK_\alpha^{-1}.}$$

▶ Follows from explicit formulas for Δ,S on both $\overline{U}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ (the f-las are the same in both cases):

$\Delta: E_\alpha\mapsto E_\alpha\otimes 1+K_\alpha\otimes E_\alpha,\ F_\alpha\mapsto F_\alpha\otimes K_\alpha^{-1}+1\otimes F_\alpha,\ K_\alpha\mapsto K_\alpha\otimes K_\alpha$

$S: E_\alpha\mapsto -K_\alpha^{-1}E_\alpha,\ F_\alpha\mapsto -F_\alpha K_\alpha,\ K_\alpha\mapsto K_\alpha^{-1}$ ■

More generally, we have:

Lemma 2: For any $\alpha\in\Pi$, $r\in\mathbb{N}$, and $x\in\overline{U}_q(\mathfrak{g})$ or $U_q(\mathfrak{g})$, we have:

$$\boxed{\begin{aligned}&(a)\ ad(E_\alpha^r)x=\sum_{i=0}^r(-1)^i q_\alpha^{i(r-1)}\begin{bmatrix} r\\ i\end{bmatrix}_\alpha E_\alpha^{r-i}K_\alpha^i x K_\alpha^{-i}E_\alpha^i\\ &(b)\ ad(F_\alpha^r)x=\sum_{i=0}^r(-1)^{r-i} q_\alpha^{-(r-i)(r-1)}\begin{bmatrix} r\\ i\end{bmatrix}_\alpha F_\alpha^i x F_\alpha^{r-i}K_\alpha^r\end{aligned}}$$

▶ Recall the homomorphisms $\iota_\alpha: U_{q_\alpha}(sl_2)\to U_q(\mathfrak{g})$ which are Hopf alg. morphisms.
On the other hand, in [Lecture 8, Lemmas 1-2], we derived explicit formulas for $\Delta(E^r)$, $\Delta(F^r)$, $S(E^r)$, $S(F^r)$. Combining these two points we get:

$*\ \Delta(E_\alpha^r)=\sum_{i=0}^r q_\alpha^{i(r-i)}\begin{bmatrix} r\\ i\end{bmatrix}_\alpha E_\alpha^{r-i}K_\alpha^i\otimes E_\alpha^i,\quad S(E_\alpha^i)=(-1)^i q_\alpha^{i(i-1)}K_\alpha^{-i}E_\alpha^i \Rightarrow$ part (a)

$*\ \Delta(F_\alpha^r)=\sum_{i=0}^r q_\alpha^{i(r-i)}\begin{bmatrix} r\\ i\end{bmatrix}_\alpha F_\alpha^i\otimes F_\alpha^{r-i}K_\alpha^{-i},\quad \left.\begin{aligned}&S(F_\alpha^{r-i})=(-1)^{r-i}q_\alpha^{-(r-i)(r-i-1)}F_\alpha^{r-i}K_\alpha^{r-i}\\ &S(K_\alpha^{-i})=K_\alpha^i\end{aligned}\right\}\Rightarrow$ part (b) ■

Recall the elements $u^\pm_{\alpha\beta}$ introduced last time $(\alpha,\beta\in\Pi,\ \alpha\neq\beta)$.

Lemma 3: For $\alpha\neq\beta\in\Pi$, we have:

$$\boxed{\begin{array}{l}(a)\ ad(E_\alpha^{1-a_{\alpha\beta}})(E_\beta)=u^+_{\alpha\beta}\\ (b)\ ad(F_\alpha^{1-a_{\alpha\beta}})(F_\beta K_\beta)=u^-_{\alpha\beta}K_\beta K_\alpha^{1-a_{\alpha\beta}}\end{array}}$$

► Applying Lemma 2(a) with $x=E_\beta$, $\tau=1-a_{\alpha\beta}$, we get:

$$ad(E_\alpha^\tau)E_\beta=\sum_{i=0}^{\tau}(-1)^i q_\alpha^{i(\tau-1)}\begin{bmatrix}\tau\\ i\end{bmatrix}_\alpha E_\alpha^{\tau-i}K_\alpha^i E_\beta K_\alpha^{-i}E_\alpha^i=\sum_{i=0}^{\tau}(-1)^i q_\alpha^{i(\tau-1)}\cdot q^{i(\alpha,\beta)}\cdot\begin{bmatrix}\tau\\ i\end{bmatrix}_\alpha\cdot E_\alpha^{\tau-i}E_\beta E_\alpha^i$$

But $q^{(\alpha,\beta)}=q^{a_{\alpha\beta}\cdot\frac{(\alpha,\alpha)}{2}}=q_\alpha^{a_{\alpha\beta}}\Rightarrow q^{i(\alpha,\beta)}=q_\alpha^{ia_{\alpha\beta}}$

$\tau=1-a_{\alpha\beta}\Rightarrow q_\alpha^{i(\tau-1)}=q_\alpha^{-ia_{\alpha\beta}}$ $\Big\}\Rightarrow q_\alpha^{i(\tau-1)}\cdot q^{i(\alpha,\beta)}=1$ $\Big\}\Rightarrow$ (a)

Likewise, applying Lemma 2(b) with $x=F_\beta K_\beta$, we get (b) ▪

Using the above viewpoint towards $u^\pm_{\alpha\beta}$ via adjoint representation of $U_q(\mathfrak{g})$, we get:

Lemma 4: For $\alpha\neq\beta\in\Pi$, we have $\forall\gamma\in\Pi$ the following equalities:

$$\boxed{[F_\gamma,u^+_{\alpha\beta}]=0\ \text{ and }\ [E_\gamma,u^-_{\alpha\beta}]=0.}$$

► We will prove the former equality (the latter is proved analogously or via Cartan involution applied to the former)

- If $\gamma\neq\alpha,\beta$, then $[F_\gamma,E_\alpha]=0=[F_\gamma,E_\beta]\Rightarrow$ obviously $[F_\gamma,u^+_{\alpha\beta}]=0$.
- If $\gamma=\beta$, then $[F_\gamma,E_\alpha]=0$, while $[F_\beta,E_\beta]=-\frac{K_\beta-K_\beta^{-1}}{q_\beta-q_\beta^{-1}}$. Set $\tau:=1-a_{\alpha\beta}$ $\Big\}\Rightarrow$

  $u^+_{\alpha\beta}=\sum_{i=0}^{\tau}(-1)^i\begin{bmatrix}\tau\\ i\end{bmatrix}_\alpha E_\alpha^{\tau-i}E_\beta E_\alpha^i$

  $$\Rightarrow[F_\beta,u^+_{\alpha\beta}]=\sum_{i=0}^{\tau}(-1)^i\begin{bmatrix}\tau\\ i\end{bmatrix}_\alpha\cdot E_\alpha^{\tau-i}\cdot\frac{K_\beta^{-1}-K_\beta}{q_\beta-q_\beta^{-1}}\cdot E_\alpha^i=\sum_{i=0}^{\tau}(-1)^i\begin{bmatrix}\tau\\ i\end{bmatrix}_\alpha\cdot q^{-i(\alpha,\beta)}E_\alpha^\tau K_\beta^{-1}\cdot\frac{1}{q_\beta-q_\beta^{-1}}-\sum_{i=0}^{\tau}(-1)^i\begin{bmatrix}\tau\\ i\end{bmatrix}_\alpha\cdot q^{i(\alpha,\beta)}E_\alpha^\tau K_\beta\cdot\frac{1}{q_\beta-q_\beta^{-1}}.$$

  Recalling the f-la $q^{i(\alpha,\beta)}=(q_\alpha^{a_{\alpha\beta}})^i$ (see proof of Lemma 3 above), it remains to apply the equalities $\boxed{\sum_{i=0}^{\tau}(-1)^i\begin{bmatrix}\tau\\ i\end{bmatrix}_\alpha(q_\alpha^{\pm(1-\tau)})^i=0}$ to see that $[F_\beta,u^+_{\alpha\beta}]=0$. ✓

- If $\gamma=\alpha$, then $[F_\gamma,E_\beta]=0$. But there are many terms E_α in $u^+_{\alpha\beta}$. Hence, we will a bit smarter argument than for $\gamma=\beta$. According to Lemma 3(a) & Lemma 1:

  $[F_\alpha,u^+_{\alpha\beta}]=ad(F_\alpha)(u^+_{\alpha\beta})\cdot K_\alpha^{-1}$, $ad(F_\alpha)(u^+_{\alpha\beta})=ad(F_\alpha E_\alpha^\tau)(E_\beta)$ with $\tau:=1-a_{\alpha\beta}$.

  Let us rewrite $F_\alpha E_\alpha^\tau$ by moving F_α to the right. We have:

  $F_\alpha E_\alpha^\tau=E_\alpha^\tau F_\alpha-[\tau]_\alpha E_\alpha^{\tau-1}[K_\alpha;\tau-1]\Rightarrow$ need to compute $ad\left(\begin{smallmatrix}\text{both}\\ \text{terms}\end{smallmatrix}\right)(E_\beta)$.

  Lemma 1 $\Rightarrow ad(F_\alpha)(E_\beta)=[F_\alpha,E_\beta]\cdot K_\alpha=0$, while $ad(K_\alpha)(E_\beta)=q^{(\alpha,\beta)}E_\beta=q_\alpha^{1-\tau}E_\beta$

  So: $ad(F_\alpha E_\alpha^\tau)(E_\beta)=-[\tau]_\alpha\,ad(E_\alpha^{\tau-1})\cdot\left(\frac{q_\alpha^{1-\tau}\cdot q_\alpha^{\tau-1}-q_\alpha^{\tau-1}q_\alpha^{1-\tau}}{q_\alpha-q_\alpha^{-1}}E_\beta\right)=0$ ✓

▪

Now we are ready to prove the triangular decomposition for $U_q(\mathfrak{g})$.

Prop 1: Let I^+ (resp. I^-) be the 2-sided ideal of $\overline{U}_q^+$ (resp. $\overline{U}_q^-$) gen-d by $u^+_{\alpha\beta}$ (resp. $u^-_{\alpha\beta}$)

(a) The 2-sided ideal in $\overline{U}_q(\mathfrak{g})$ generated by all $u^+_{\alpha\beta}$ is equal to the image $\mathrm{mult}(\overline{U}_q^- \otimes \overline{U}_q^0 \otimes I^+) =: \widetilde{I}^+$

(b) The 2-sided ideal in $\overline{U}_q(\mathfrak{g})$ generated by all $u^-_{\alpha\beta}$ is equal to the image $\mathrm{mult}(I^- \otimes \overline{U}_q^0 \otimes \overline{U}_q^+) =: \widetilde{I}^-$.

▶ (a) It suffices to prove that $\widetilde{I}^+$ is a 2-sided ideal of $\overline{U}_q(\mathfrak{g})$. As a vector space $\widetilde{I}^+$ is spanned by all $\{x \cdot u^+_{\alpha\beta} \cdot E_I \mid x \in \overline{U}_q(\mathfrak{g}),\ I\text{-seq. of simple roots}\}$. In particular, it is clear that $\widetilde{I}^+$ is a left ideal of $\overline{U}_q(\mathfrak{g})$.

- For $\gamma \in \Pi$, we also have $(x \cdot u^+_{\alpha\beta} \cdot E_I) E_\gamma = x \cdot u^+_{\alpha\beta} \cdot E_{(I,\gamma)} \in \widetilde{I}^+$
- For $\lambda \in Q$, we have $(x \cdot u^+_{\alpha\beta} \cdot E_I) K_\lambda = q^{-(\lambda, wt I) - (\lambda, \beta + (1-a_{\alpha\beta})\alpha)} \cdot x K_\lambda \cdot u^+_{\alpha\beta} \cdot E_I \in \widetilde{I}^+$

Finally:

- For $\gamma \in \Pi$, we have $(x \cdot u^+_{\alpha\beta} \cdot E_I) F_\gamma = \underbrace{x F_\gamma \cdot u^+_{\alpha\beta} \cdot E_I}_{\in \widetilde{I}^+} - x \cdot \underbrace{[F_\gamma, u^+_{\alpha\beta}]}_{= 0 \text{ by Lemma 4}} E_I - x \cdot u^+_{\alpha\beta} \cdot [F_\gamma, E_I]$.

As $[F_\gamma, E_I] \in \overline{U}_q^{\geq}$ (subalg. gen-d by $E_\alpha, K_\alpha^{\pm}$), the last term is also an el-t of $\widetilde{I}^+$

(b) Follows in the same way or can be deduced from (a) by applying the antiautomorphism $\sigma \circ \omega$ of $\overline{U}_q(\mathfrak{g})$ ■

Thm 1 (Triangular decomposition for $U_q(\mathfrak{g})$):

(a) The multiplication map $U_q^- \otimes U_q^0 \otimes U_q^+ \to U_q(\mathfrak{g})$ is an isomorphism of v.spaces

(b) The algebra U_q^- is isom. to the algebra gen-d by $\{F_\alpha\}_{\alpha \in \Pi}$ subject to the q-Serre rel-s b/w them.

(c) The algebra U_q^+ is isom. to the algebra gen-d by $\{E_\alpha\}_{\alpha \in \Pi}$ subject to the q-Serre rel-s b/w them.

(d) The el-s $\{K_\lambda\}_{\lambda \in Q}$ form a k-basis of U_q^0.

Cor: (a) The multiplication map $U_q^+ \otimes U_q^0 \otimes U_q^- \to U_q(\mathfrak{g})$ is also an isom. of v-spaces (just apply ω to Thm 1)

(b) The multiplication maps $U_q^0 \otimes U_q^+ \to U_q^{\geq}$ (defined above), $U_q^- \otimes U_q^0 \to U_q^{\leq}$ (subalg gen-d by $F_\alpha, K_\alpha^{\pm 1}$) are isom. of vector spaces.

(c) For degree reasons, the images of $\{E_\alpha^r, F_\alpha^r \mid \alpha \in \Pi, r \in \mathbb{Z}_{>0}\}$ in $U_q(\mathfrak{g})$ are lin. ind.

Cor: The homomorphisms $l_\alpha : U_{q_\alpha}(\mathfrak{sl}_2) \to U_q(\mathfrak{g})$ are injective.

(Proof of Thm 1).

The kernel $\mathcal{I}$ of the canonical projection $\pi: \bar{U}_q(\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g})$ is generated by $\{U^{\pm}_{\alpha\beta}\}$. Due to Prop 1, it equals $m(I^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ + \bar{U}_q^- \otimes \bar{U}_q^0 \otimes I^+)$, where m = multiplicat.

- As $(I^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ + \bar{U}_q^- \otimes \bar{U}_q^0 \otimes I^+) \cap (k \otimes \bar{U}_q^0 \otimes k) = 0$, we get $\mathcal{I} \cap \bar{U}_q^0 = 0$, due to the triangular decomposition of $\bar{U}_q(\mathfrak{g})$. Hence, $\pi: \boxed{\bar{U}_q^0 \xrightarrow{\sim} U_q^0}$ - isom. of v.spaces.
- As $(I^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ + \bar{U}_q^- \otimes \bar{U}_q^0 \otimes I^+) \cap (k \otimes k \otimes \bar{U}_q^+) = k \otimes k \otimes I^+$, we get $\mathcal{I} \cap \bar{U}_q^+ = I^+$, due to triang. decomp. of $\bar{U}_q(\mathfrak{g})$. Hence, π induces an isomorphism $\boxed{\bar{U}_q^+/I^+ \xrightarrow{\sim} U_q^+}$. As $\bar{U}_q^+$ is a free algebra in $\{E_\alpha\}_{\alpha\in\Pi}$, we get part (c).
- Completely analogously, we have $\boxed{\bar{U}_q^-/I^- \xrightarrow{\sim} U_q^-}$ $\Rightarrow$ part (b).

Finally, we note that $(\bar{U}_q^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+)/(I^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ + \bar{U}_q^- \otimes \bar{U}_q^0 \otimes I^+)$ is canonically identified with $(\bar{U}_q^-/I^-) \otimes \bar{U}_q^0 \otimes (\bar{U}_q^+/I^+)$. Combining this observation with the triangular decomposition of $\bar{U}_q(\mathfrak{g})$ and the above three isomorphisms imply part (a) of Thm 1

Next, we shall study finite-dimensional $U_q(\mathfrak{g})$-modules for $q \neq \sqrt{1}$.

Recall:
- For $\mathfrak{g} = \mathfrak{sl}_2$, we saw that any fin. dim. $U_q(\mathfrak{sl}_2)$-modules is semisimple and the simple f.d. modules are parametrized by $\mathbb{N} \times \{\pm\}$.
- In the classical story, all the embedded $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ give a powerful tool to study $\mathrm{Rep}_{f.d.}\,\mathfrak{g}$ knowing it for $\mathfrak{sl}_2$-case.

Recall that P denotes the weight lattice (that is, $(\lambda,\mu) \in \mathbb{Z}$ $\forall \lambda \in P, \mu \in Q$). Let $\varsigma: Q \to \{\pm 1\}$ be a group homomorphism (equivalently, we assign $+1$ or -1 to each positive simple root) and $\lambda \in P$. For any $U_q(\mathfrak{g})$-module M, define

$$\boxed{M_{\lambda,\varsigma} := \{m \in M \mid K_\mu(m) = \varsigma(\mu)\cdot q^{(\lambda,\mu)} m \ \ \forall \mu \in Q\}} \text{ - subspace of } M.$$

Lemma 5: Let M be a fin. dim. $U_q(\mathfrak{g})$-module. Then:

(a) $M = \bigoplus_{(\lambda,\varsigma)} M_{\lambda,\varsigma}$ (assuming $\mathrm{char}(k) \neq 2$)

(b) $E_\alpha M_{\lambda,\varsigma} \subset M_{\lambda+\alpha,\varsigma}$, $F_\alpha M_{\lambda,\varsigma} \subset M_{\lambda-\alpha,\varsigma}$ $\forall \alpha \in \Pi$

(c) $\{E_\alpha, F_\alpha\}_{\alpha\in\Pi}$ act nilpotently on M.