▸ (Proof of Lemma 5 from last time

- First, we note that $q \neq \sqrt{1} \Rightarrow$ the sum of $M_{\lambda,\varsigma}$ is indeed a direct sum.
- Second, the equalities $K_\mu E_\alpha K_\mu^{-1} = q^{(\mu,\alpha)}$, $K_\mu F_\alpha K_\mu^{-1} = q^{-(\mu,\alpha)} F_\alpha$ imply part (b)
- To prove (c), note that M becomes an $U_{q_\alpha}(\mathfrak{sl}_2)$-module via the homom. $\iota_\alpha : U_{q_\alpha}(\mathfrak{sl}_2) \to U_q(\mathfrak{g})$. On the other hand, it was proved in [Lecture 5, Lemma 2] that E, F act nilpotently on any fin. dim. $U_{q_\alpha}(\mathfrak{sl}_2)$-module, hence, (c).
- Likewise, using ι_α and appealing to [Lecture 5, Lemma 3], the action of each K_α $(\alpha\in\Pi)$ is diagonalizable in M. As $\{K_\alpha\}_{\alpha\in\Pi}$ pair-wise commute, we can diagonalize all of them simultaneously on M.

  If $m \in M$ is a common eigenvector for all K_α $(\alpha\in\Pi)$, then $\exists\, \varepsilon_\alpha \in \{\pm1\}, a_\alpha \in \mathbb{Z}$, s.t. $K_\alpha(m) = \varepsilon_\alpha \cdot q_\alpha^{a_\alpha} \cdot m$ $\forall \alpha\in\Pi$. Set $\lambda := \sum_{\alpha\in\Pi} a_\alpha \cdot \omega_\alpha$ (ω_α - fund. weight), ς determined via $\varsigma(\alpha) = \varepsilon_\alpha$. As $(\lambda,\alpha) = a_\alpha \cdot d_\alpha \Rightarrow q^{(\lambda,\alpha)} = q_\alpha^{a_\alpha}$, we see that $\boxed{m \in M_{\lambda,\varsigma}}$

From now, we assume $\mathrm{char}(k) \neq 2$

Corollary: Given a fin. dim. $U_q(\mathfrak{g})$-module M, it splits into a direct sum $M = \bigoplus_\varsigma M^\varsigma$ over all homom-s $\varsigma : Q \to \{\pm1\}$, where $M^\varsigma := \bigoplus_{\lambda\in P} M_{\lambda,\varsigma}$.

Def: M is of type ς if $M = M^\varsigma$.

Rmk: (1) The category of fin. dim. $U_q(\mathfrak{g})$-modules is the direct sum (over all ς) of the categories of fin. dim. $U_q(\mathfrak{g})$-modules of type ς.

(2) Twisting a repr. of type $\mathbb{1}$ (i.e. $\varsigma(\lambda)=1$ $\forall \lambda\in Q$) by an algebra automorphism $\tilde{\varsigma}$ of $U_q(\mathfrak{g})$ given by $E_\alpha \mapsto \varsigma(\alpha) E_\alpha$, $F_\alpha \mapsto F_\alpha$, $K_\alpha \mapsto \varsigma(\alpha) K_\alpha$ yields an equivalence of categories b/w the category of all fin. dim. $U_q(\mathfrak{g})$-modules of type $\mathbb{1}$ and those of type ς.

(3) For any $\lambda, \lambda', \varsigma, \varsigma'$, we have $M_{\lambda,\varsigma} \otimes N_{\lambda',\varsigma'} \subset (M\otimes N)_{\lambda+\lambda', \varsigma\varsigma'}$. In particular, if M, N - $U_q(\mathfrak{g})$-modules of type $\mathbb{1}$, then $M\otimes N$ is also an $U_q(\mathfrak{g})$-mod. of type $\mathbb{1}$.

(4) All 1-dim $U_q(\mathfrak{g})$-modules are parametrized by ς as above, let $L(0,\varsigma)$ denote the corr. module. Then $\tilde{\varsigma}$-twist of a type $\mathbb{1}$ module M is isomorphic to $L(0,\varsigma)\otimes M$.

(5) If M is of type ς, then its dual M^* is also of type ς, since $(M^*)_{\lambda,\varsigma} = (M_{-\lambda,\varsigma})^*$.

! From now on, we will be working only with type $\mathbb{1}$ fin. dim. $U_q(\mathfrak{g})$-mod-s. (according to the previous Rmk, we do not loose any essential information).

Set $M_\lambda := M_{\lambda,\mathbb{1}}$, so that $M = \bigoplus_{\lambda\in P} M_\lambda$.

Lemma 1: Let M be a fin. dim. $U_q(\mathfrak{g})$-module, $M \neq 0$.

(a) $\exists\, \lambda \in P$, $m \in M_\lambda \setminus \{0\}$ s.t. $E_\alpha(m) = 0$ $\forall \alpha \in \Pi$.

(b) Let $\lambda \in P$, $m \in M_\lambda \setminus \{0\}$, s.t. $E_\alpha(m) = 0$ $\forall \alpha\in\Pi$. Then λ is a dominant weight.
Moreover, $F_\alpha^{a(\alpha)+1}(m) = 0$ $\forall \alpha \in \Pi$, where $a(\alpha) := \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$.

► (a) As $M = \bigoplus_\lambda M_\lambda$, $\dim M < \infty \Rightarrow \exists \lambda$ s.t. $M_\lambda \neq 0$, $M_{\lambda'} = 0$ $\forall \lambda' > \lambda$. Hence $E_\alpha M_\lambda \subset M_{\lambda+\alpha} = 0$.

(b) Consider M as a $U_{q_\alpha}(\mathfrak{sl}_2)$-module (via $\iota_\alpha : U_{q_\alpha}(\mathfrak{sl}_2) \longrightarrow U_q(\mathfrak{g})$).
Know: $E_\alpha(m) = 0$, $K_\alpha(m) = q^{(\lambda,\alpha)}\cdot m = q^{d_\alpha \cdot a(\alpha)} \cdot m = q_\alpha^{a(\alpha)} \cdot m$
Hence, the $U_{q_\alpha}(\mathfrak{sl}_2)$-submodule of M generated by m is an image (under $U_{q_\alpha}(\mathfrak{sl}_2)$-morphism) of the $U_{q_\alpha}(\mathfrak{sl}_2)$-Verma module $M(q_\alpha^{a(\alpha)})$, but also is fin. dim. Therefore, due to [Lecture 7, Lemma 1], $a(\alpha) \in \mathbb{Z}_{\geq 0}$ and this submodule is isomorphic to $L(a(\alpha),+)$. But the structure of the latter also implies $F_\alpha^{a(\alpha)+1}(m) = 0$ ∎

Similarly to the classical case and the case of $U_q(\mathfrak{g})$ an important role is played by the Verma modules. Fix $\lambda\in P$ and consider the 1-dim $U_q^{\geq}$-module k_λ, where each E_α acts by 0, while K_α acts as multiplication by $q^{(\lambda,\alpha)}$.

‖ Def: The $U_q(\mathfrak{g})$-Verma-module of highest weight $\lambda \in P$ is defined as

$$\boxed{M(\lambda) := \mathrm{Ind}_{U_q^{\geq}}^{U_q(\mathfrak{g})}(k_\lambda)}$$

Explicitly, $M(\lambda) = U_q(\mathfrak{g})/J_\lambda$, where J_λ is the left ideal of $U_q(\mathfrak{g})$ generated by $\{E_\alpha, K_\alpha - q^{(\lambda,\alpha)}\}_{\alpha\in\Pi}$. Denote the coset of 1 by v_λ. Then $E_\alpha(v_\lambda) = 0$, $K_\alpha(v_\lambda) = q^{(\lambda,\alpha)} \cdot v_\lambda$ $\forall \alpha \in \Pi$.
As always, $M(\lambda)$ has the following UP (universal property):

Lemma 2: If M is an $U_q(\mathfrak{g})$-module and $m \in M_\lambda$ s.t. $E_\alpha(m) = 0$ $\forall \alpha\in\Pi$, then there is a unique $U_q(\mathfrak{g})$-morphism $M(\lambda) \longrightarrow M$ s.t. $v_\lambda \mapsto m$.

► Obvious ∎

Rmk: Due to the triangular decomposition of $U_q(\mathfrak{g})$, the map $U_q^- \to M(\lambda)$ given by $x \mapsto x(v_\lambda)$ is an isomorphism of vector spaces. In particular, $(M(\lambda))_{\lambda - n\alpha} = k\cdot F_\alpha^n(v_\lambda)$ $\forall \alpha\in\Pi$, $n\in \mathbb{Z}_{\geq 0}$

As in the classical case, $M(\lambda)$ admits a unique maximal submodule ($\neq M(\lambda)$) and therefore $M(\lambda)$ has a unique simple factor module, denoted $L(\lambda)$.

The following is a q-counterpart of the classical result:

Thm 1: For each dominant $\lambda \in P$ (i.e. $(\lambda,\alpha)\geq 0\ \forall \alpha\in\Pi$), the simple $U_q(\mathfrak{g})$-module $L(\lambda)$ is finite-dimensional. Vice-versa, each fin. dim. simple $U_q(\mathfrak{g})$-module is isomorphic to exactly one $L(\lambda)$ with $\lambda\in P$-dominant.

The proof is based on the following few results.

Lemma 3: Let $\lambda\in P$, $\alpha\in\Pi$ satisfy $(\lambda,\alpha)\geq 0$, set $n := \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} = \frac{(\lambda,\alpha)}{d_\alpha}$.

There is a homom. of $U_q(\mathfrak{g})$-modules $\varphi: M(\lambda-(n+1)\alpha) \to M(\lambda)$,
$$v_{\lambda-(n+1)\alpha} \longmapsto F_\alpha^{n+1}(v_\lambda)$$

▶ Clearly $F_\alpha^{n+1}(v_\lambda)\in M(\lambda)_{\lambda-(n+1)\alpha}$. Hence, due to UP of Verma modules, it suffices to show $E_\beta(F_\alpha^{n+1}(v_\lambda))=0\ \forall\beta\in\Pi$. If $\beta\neq\alpha$, then $E_\beta F_\alpha = F_\alpha E_\beta \Rightarrow E_\beta F_\alpha^{n+1}v_\lambda = F_\alpha^{n+1}E_\beta v_\lambda = 0$.
On the other hand, for $\beta=\alpha$ we will apply the equality
$$E_\alpha F_\alpha^{n+1} = F_\alpha^{n+1}E_\alpha + [n+1]_\alpha\cdot F_\alpha^{n}\cdot [K_\alpha;-n]$$
together with $E_\alpha(v_\lambda) = [K_\alpha;-n](v_\lambda)=0$.
Alternatively, for $\beta=\alpha$, one can appeal to the case of $U_{q_\alpha}(\mathfrak{sl}_2)$ treated before. ■

Def: An operator $A: V \circlearrowleft$ is *locally nilpotent* if $\forall v\in V\ \exists n\in\mathbb{N}$ s.t. $A^n(v)=0$.

Lemma 4: Let V be a $U_q(\mathfrak{sl}_2)$-module s.t. $V=\bigoplus_{n\in\mathbb{Z}} V_n$ with $V_n=\{v\in V \mid K(v)=q^n\cdot v\}$. Suppose E,F act locally nilpotently on V and $\dim V_n<\infty\ \forall n$. Then V is fin. dim. and $\dim V_n=\dim V_{-n}\ \forall n$.

▶ The proof is by induction on $\dim V_0+\dim V_1$.
Pick $l\in\mathbb{Z}$ and $w\in V_l\setminus\{0\}$. As E acts nilpotently $\exists r\in\mathbb{Z}_{>0}$ s.t. $E^{r-1}(w)\neq 0$, $E^r(w)=0$.
Set $v:=E^{r-1}(w)$, so that $E(v)=0$ and $v\in V_{n=l+2(r-1)}$. Hence, the submodule $U_q(\mathfrak{sl}_2)(v)$ is an image of $M(q^n)$. However, $F^{\gg 0}(v)=0 \Rightarrow U_q(\mathfrak{sl}_2)(v)\simeq L(n,+)$. But $\forall n\in\mathbb{N}$: $\dim L(n,+)_0+\dim L(n,+)_1=1$. Hence, the induction can be applied to the factor module $V':=V/U_q(\mathfrak{sl}_2)(v)$. The last equality follows from $\dim L(n,+)_a=\dim L(n,+)_{-a}$ ■

Lemma 5: Let $\lambda\in P$. Choose $m(\alpha), n(\alpha)\in\mathbb{Z}_{>0}\ \forall\alpha\in\Pi$. Let I be the left ideal of $U_q(\mathfrak{g})$ generated by $\{E_\alpha^{m(\alpha)}, F_\alpha^{n(\alpha)}, K_\alpha-q^{(\lambda,\alpha)}\}$. Then all $\{E_\alpha,F_\alpha\}_{\alpha\in\Pi}$ act locally nilpotently on the $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})/I$.

Exercise: Prove Lemma 5.

Now we are ready to prove Theorem 1.

▶ (Proof of Theorem 1)

• If M is a fin. dim. simple $U_q(\mathfrak{g})$-module, then by Lemma 1(a) $\exists\, \lambda \in P$, $m \in M_\lambda \setminus \{0\}$ s.t. $E_\alpha(v)=0$ $\forall \alpha \in \Pi$. By UP of Verma modules, there is a $U_q(\mathfrak{g})$-morphism $M(\lambda) \longrightarrow M$ sending $v_\lambda \longmapsto m$. Since M is simple, we have $M \simeq L(\lambda)$. Moreover, λ is unique as it is the maximal weight. But due to Lemma 1(b), λ-dominant

This completes the proof of a simple half of Thm 1.

• Let us now pick a dominant $\lambda \in P$ and consider the corresponding simple $L(\lambda)$. Set $n(\alpha) := \frac{(\lambda,\alpha)}{d_\alpha} = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$ $\forall \alpha \in \Pi$. According to Lemma 3, we have $U_q(\mathfrak{g})$-morphisms $\varphi_\alpha : M(\lambda-(n(\alpha)+1)\alpha) \longrightarrow M_\lambda$, sending $v_{\lambda-(n(\alpha)+1)\alpha} \mapsto F_\alpha^{n(\alpha)+1}(v_\lambda)$. Hence, the result follows from the following claim:

CLAIM: The $U_q(\mathfrak{g})$-module $\boxed{\tilde{L}(\lambda) := M(\lambda)/\sum_{\alpha\in\Pi} \mathrm{Im}(\varphi_\alpha)}$ is finite dimensional.

To prove this claim, we note first that

$$\boxed{\tilde{L}(\lambda) \cong U_q(\mathfrak{g}) \Big/ \Big(\sum_{\alpha\in\Pi} U_q(\mathfrak{g})\cdot E_\alpha + \sum_{\alpha\in\Pi} U_q(\mathfrak{g}) F_\alpha^{n(\alpha)+1} + \sum_{\alpha\in\Pi} U_q(\mathfrak{g})(K_\alpha - q^{(\lambda,\alpha)})\Big)}$$

According to Lemma 5, all $\{E_\alpha, F_\alpha\}$ act loc. nilpotently on $\tilde{L}(\lambda)$.
The rest of the argument will be parallel to the classical setup, i.e. we will verify that the set of weights of $\tilde{L}(\lambda)$ is W-stable (W = Weyl gp). This would imply $\dim \tilde{L}(\lambda) < \infty$, since each W-orbit contains a dominant weight, the number of dominant weights less or equal to λ is finite, and all weight spaces are f.d.

Suffices to show that if $\boxed{\tilde{L}(\lambda)_\mu \neq 0 \Rightarrow \tilde{L}(\lambda)_{s_\alpha(\mu)} \neq 0}$ (recall that for $\alpha \in \Pi, \mu \in P$: $s_\alpha(\mu) = \mu - \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha$)

Consider $V := \bigoplus_{n\in\mathbb{Z}} \tilde{L}(\lambda)_{\mu+n\alpha}$. Viewing $U_{q_\alpha}(\mathfrak{sl}_2) \curvearrowright \tilde{L}(\lambda)$ via $\iota_\alpha : U_{q_\alpha}(\mathfrak{sl}_2) \longrightarrow U_q(\mathfrak{g})$, V is a $U_{q_\alpha}(\mathfrak{sl}_2)$-submodule. Moreover, by above E, F act lo. nilpotently on V, while $V = \bigoplus_{m\in\mathbb{Z}} V_m$ with $V_m = \tilde{L}(\lambda)_{\mu+n\alpha}$ for $m = 2n + \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$. Therefore, we are in the setup of Lemma 4 $\Rightarrow \dim V_r = \dim V_{-r}$.

Set $r := \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$ so that $V_r \neq 0 \Rightarrow V_{-r} \neq 0 \Rightarrow \tilde{L}(\lambda)_{s_\alpha(\mu)} \neq 0$.

This completes the proof of Claim, hence, also Thm 1. ∎

Cor: For any f.d. $U_q(\mathfrak{g})$-module M, $\boxed{\dim M_\mu = \dim M_{w(\mu)}}$ $\forall\, \mu \in P, w \in W$.

Remk: Next time we will see that $\tilde{L}(\lambda) \cong L(\lambda)$.

The following result is quite useful in applications.

<u>Thm 2</u>: Let $u \in U_q(\mathfrak{g})$ be an element acting trivially on all fin. dim. $U_q(\mathfrak{g})$-reps. Then $u=0$.

▸ • Recall the $U_q(\mathfrak{g})$-modules $\tilde{L}(\lambda)$ from the proof of Thm 1 for every dominant $\lambda \in P$. That is $\tilde{L}(\lambda) = M(\lambda)/\sum_{\alpha\in\Pi} \mathrm{Im}(\varphi_\alpha)$. Note that while for every $\nu \in Q_+$, the natural map (given by an action on the image of v_λ) $(U_q^-)_{-\nu} \to \tilde{L}(\lambda)_{\lambda-\nu}$ is surjective, it is also injective for "small enough" ν. To be precise, let us write $\lambda = \sum_{\alpha\in\Pi} n_\alpha \cdot \omega_\alpha$ $(n_\alpha \in \mathbb{Z}_{\geq 0})$, $\nu = \sum_{\alpha\in\Pi} m_\alpha \cdot \alpha$ $(m_\alpha \in \mathbb{Z}_{\geq 0})$, in particular, n_α is consistent with n from Lemma 3, which was picked to construct embeddings $\varphi_\alpha: M(\lambda-(n+1)\alpha) \to M(\lambda)$. Thus, if $\boxed{0\leq m_\alpha \leq n_\alpha \ \forall \alpha\in\Pi}$ then $\left(\sum_\alpha \mathrm{Im}(\varphi_\alpha)\right)_{\lambda-\nu} = 0 \Rightarrow$ the map $(U_q^-)_{-\nu} \to \tilde{L}(\lambda)_{\lambda-\nu}$ is isom. of vector spaces.

• We also have "Cartan twisted" modules $\tilde{L}(\lambda)^\omega$, obtained from $\tilde{L}(\lambda)$ via a <u>twist</u> by the Cartan involution. In particular, the vector v'_λ (we put $'$ to distinguish) of $\tilde{L}(\lambda)^\omega$ satisfies $\boxed{F_\alpha(v'_\lambda)=0 \ \forall \alpha\in\Pi, \ K_\mu(v'_\lambda) = q^{-(\mu,\lambda)}\cdot v'_\lambda \ \forall \mu\in Q}$ Due to the above observation, the map $(U_q^+)_\nu \to (\tilde{L}(\lambda)^\omega)_{-\lambda+\nu}$ is isom. if $\nu = \sum_\alpha m_\alpha \cdot \alpha$ with $0 \leq m_\alpha \leq n_\alpha$.

Similarly to the proof of triangular decomp. for $\bar{U}_q(\mathfrak{g})$ [Lecture 10, Thm 3], we are gonna consider $U_q(\mathfrak{g})$-action on $\tilde{L}(\lambda)\otimes\tilde{L}(\lambda')$, which is fin. dim. We shall see that if $u(v_\lambda \otimes v'_{\lambda'}) = 0 \ \forall \lambda, \lambda'$-dominant, then u <u>must be zero</u>.

• Write u in the form $u = \sum_{i,j,\mu} a_{j,\mu,i}\cdot y_j \cdot K_\mu \cdot x_i$, where $\{x_i\}_i$ – basis of U_q^+, $\{y_j\}_j$ – basis of U_q^-.
We will refer to weights of x_i, y_j by $wt(i)$, $-wt(j)$.
Pick $\nu \in Q$ such that $\exists i,j,\mu$ with $wt(i)=\nu$, $a_{j,\mu,i}\neq 0$, and ν being maximal with this property.
Note that $\Delta(x_i) = K_{wt(i)}\otimes x_i + \underbrace{(\text{some other terms belonging to } (U_q^{\geq})_{>0}\otimes U_q^+)}_{\text{these el-s act trivially on } v_\lambda\otimes v'_{\lambda'}}$

$\Rightarrow \boxed{x_i(v_\lambda\otimes v'_{\lambda'}) = q^{(wt(i),\lambda)}\cdot v_\lambda\otimes x_i(v'_{\lambda'})}$

Acting further by K_μ, we get $\boxed{K_\mu x_i(v_\lambda\otimes v'_{\lambda'}) = q^{(wt(i),\lambda)+(\mu,\lambda-\lambda'+wt(i))}\cdot v_\lambda\otimes x_i(v'_{\lambda'})}$

Finally, when applying y_j, we recall that $\Delta(y_j) = y_j\otimes K^{-1}_{wt(j)} + \left(\begin{smallmatrix}\text{some terms belonging to}\\ U_q^-\otimes(U_q^{\leq})_{<0}\end{smallmatrix}\right)$
The maximality of ν chosen above implies that the image $y_j K_\mu x_i(v_\lambda\otimes v'_{\lambda'})$ has the second component of weight $-\lambda'+\nu$ only if $wt(i)=\nu$, and the corresponding term equals $\boxed{q^{(\underset{\nu}{wt(i)},\lambda)+(\mu,\lambda-\lambda'+\underset{\nu}{wt(i)})-(wt(j),-\lambda'+\underset{\nu}{wt(i)})}\cdot(y_j v_\lambda)\otimes(x_i v'_{\lambda'})}$

Now, picking λ,λ' "big enough" (see above explanations), one immediately gets $a_{j,\mu,i}=0$ for all i with $wt(i)=\nu$ (here we use the first two observations in the proof) ∎