— LECTURE 13 — (03/05/2018)

Last time: ① $L(\lambda)$ – fin. dim. $\Longleftrightarrow$ λ – dominant (and any simple f.d. is $\simeq L(\lambda)$)

② If $u \in U_q(\mathfrak{g})$ acts trivially on all fin. dim. $U_q(\mathfrak{g})$-modules $\Rightarrow u=0$

Our first goal for today is to prove the following two results:

Thm 1: If $\mathrm{char}(k)=0$ and q is transcendental over $\mathbb{Q}$, then for any dominant weight $\lambda \in P$, we have $\tilde{L}(\lambda)=L(\lambda)$. Moreover, the dimensions of the weight spaces $L(\lambda)_\mu$ are the same as for highest weight λ simple $\mathfrak{g}$-module.

Thm 2: If $\mathrm{char}(k)=0$ and q is transcendental over $\mathbb{Q}$, then any finite dimensional $U_q(\mathfrak{g})$-module is semisimple.

! Both theorems can be generalized to arbitrary k and $q \neq \sqrt{1}$. Later on, we will get some more tools for that.

To prove both results, we shall work over $\mathbb{Q}(v)$ first.
Let v be an indeterminate $\rightsquigarrow$ set $\boxed{\begin{array}{l} k:=\mathbb{Q}(v), \text{ pick } q:=v\in k. \\ A:=\mathbb{Q}[v,v^{-1}]. \end{array}}$

Fix a dominant weight $\lambda \in P$ and consider $L(\lambda)$, $\tilde{L}(\lambda)$. Let V be either $L(\lambda)$ or $\tilde{L}(\lambda)$.
The image of $v_\lambda \in M(\lambda)$ in V is also denoted by v_λ; $V=U_q^-(v_\lambda)$, i.e., V is spanned by $\{F_I(v_\lambda)\}_{I\text{-sequence of simple roots}}$. Moreover, as $\dim V<\infty$, there are finitely many I s.t. $F_I(v_\lambda)\neq 0$. Set $\boxed{V_A:=\sum_I A\cdot F_I(v_\lambda),\ V_{\mu,A}:=\sum_{\mathrm{wt}\, I=\lambda-\mu} A\cdot F_I(v_\lambda)}$ $\Leftarrow$ note that $V_A=\bigoplus_{\mu\in P} V_{\mu,A}$.

Exercise 1: Show that V_A, $V_{\mu,A}$ are free A-modules and the natural maps $V_{\mu,A}\otimes_A k \longrightarrow V_\mu$, $V_A\otimes_A k\longrightarrow V$ are isomorphisms.

This exercise implies $\boxed{\mathrm{rk}_A V_{\mu,A}=\dim_k V_\mu}$ $\forall \mu\in P$.

Lemma 1: The A-module V_A is stable under all $E_\alpha, F_\alpha, K_\alpha^{\pm 1}, [K_\alpha;a]$ for $\alpha\in\Pi$, $a\in\mathbb{Z}$

- Stability under F_α follows from $F_\alpha(F_I(v_\lambda))=F_{(\alpha,I)}(v_\lambda)$.
- Stability under $K_\alpha^{\pm1}$ follows from $K_\alpha^{\pm1}(F_I(v_\lambda))=q^{\pm\langle\lambda-\mathrm{wt}\, I,\alpha\rangle}\cdot F_I(v_\lambda)$
- Stability under $[K_\alpha;a]$ follows from $[K_\alpha;a](F_I(v_\lambda))=\frac{K_\alpha q_\alpha^a-K_\alpha^{-1}q_\alpha^{-a}}{q_\alpha-q_\alpha^{-1}}(F_I(v_\lambda))=$
$=[a+m]_\alpha\cdot F_I(v_\lambda)$, where $m:=\frac{2(\lambda-\mathrm{wt}(I),\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}$. Remains to use $[a+m]_\alpha\in A$.
- Stability under E_α follows from $[E_\alpha,F_\beta]=\delta_{\alpha\beta}[K_\alpha;0]$ and the previous stability ∎

Let us now "specialize $v \mapsto 1$", i.e., we consider the $\mathbb{Q}$-algebra homomorphism $\varphi: A=\mathbb{Q}[v,v^{-1}] \longrightarrow \mathbb{C}$ given by $\varphi(v)=1$.

Set $\boxed{\overline{V} := V_A \underset{A}{\otimes} \mathbb{C}\,,\ \overline{V}_\mu := V_{\mu,A} \underset{A}{\otimes} \mathbb{C}}$ Clearly, $\overline{V} = \bigoplus_{\mu\in Q} \overline{V}_\mu$.

Moreover, due to Exercise 1, we see that $\dim_{\mathbb{C}} \overline{V}_\mu = \mathrm{rk}_A V_{\mu,A} = \dim_k V_\mu$.

According to Lemma 1: $E_\alpha, F_\alpha, K_\alpha, [K_\alpha;0] \curvearrowright V_A$, hence, they induce the operators $\boxed{e_\alpha, f_\alpha, k_\alpha, h_\alpha \curvearrowright \overline{V}}$

Lemma 2: The endomorphisms $\{e_\alpha, f_\alpha, h_\alpha\}_{\alpha\in\Pi}$ give rise to an action $\mathfrak{g} \curvearrowright \overline{V}$, while all $\{k_\alpha\}_{\alpha\in\Pi}$ act by identity.

▶ • For $v \in V_{\mu,A}$, we have $k_\alpha(v\otimes 1) = (q^{(\mu,\alpha)}v)\otimes 1 = \varphi(v^{(\mu,\alpha)})\cdot v\otimes 1 = v\otimes 1$, which proves the second statement.

• Likewise, we have $h_\alpha(v\otimes 1) = ([m]_\alpha \cdot v \otimes 1) = \varphi([m]_\alpha)\cdot v\otimes 1$, where $m := \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$.

But $[m]_{q_\alpha} = \begin{cases} q_\alpha^{m-1}+q_\alpha^{m-3}+\ldots+q_\alpha^{1-m}, & m\in\mathbb{Z}_{>0} \\ -q_\alpha^{-m-1}-q_\alpha^{-m-3}-\ldots-q_\alpha^{1+m}, & m\in\mathbb{Z}_{<0} \\ 0, & m=0 \end{cases} \Rightarrow \varphi([m]_\alpha)=m.$

So: $\boxed{h_\alpha(v\otimes 1) = \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\cdot v\otimes 1 \quad \text{for } v\in V_{\mu,A}} \quad (*)$

In particular, we see that $\{h_\alpha\}_{\alpha\in\Pi}$ pairwise commute.

• The relation $[E_\alpha, F_\beta] = \delta_{\alpha\beta}\cdot[K_\alpha;0]$ immediately implies $\boxed{[e_\alpha, f_\beta] = \delta_{\alpha\beta}\cdot h_\alpha}$

• As $e_\beta: \overline{V}_\mu \longrightarrow \overline{V}_{\mu+\beta}$, we get $\boxed{[h_\alpha, e_\beta] = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} e_\beta}$ due to $(*)$.

• As $f_\beta: \overline{V}_\mu \longrightarrow \overline{V}_{\mu-\beta}$, we get $\boxed{[h_\alpha, f_\beta] = -\frac{2(\beta,\alpha)}{(\alpha,\alpha)} f_\beta}$ due to $(*)$

• Finally, since $\varphi\left(\begin{bmatrix}1-a_{\alpha\beta}\\ i\end{bmatrix}_\alpha\right) = \binom{1-a_{\alpha\beta}}{i}$, the (q)Serre relations in $U_q(\mathfrak{g})$ imply the usual Serre rel-s among $\{e_\alpha\}$ and $\{f_\alpha\}$.

Thus, we have verified that $\{e_\alpha, f_\alpha, h_\alpha\}$ satisfy all the defining rel-s $\Rightarrow$ give rise to an action $\mathfrak{g} \curvearrowright \overline{V}$. ∎

Lemma 3: $\overline{V}$ is a simple $\mathfrak{g}$-module of highest weight λ and $\overline{V}_\mu$ is the weight μ subspace of $\overline{V}$.

▶ Obviously $e_\alpha(v_\lambda\otimes 1)=0$ $(\forall \alpha\in\Pi)$ and $F_I(v_\lambda)\otimes 1 = f_{\beta_1}\ldots f_{\beta_r}(v_\lambda\otimes 1)$ if $I=(\beta_1,\ldots,\beta_r)$

Thus: $\overline{V}$ is a highest weight module and $\dim_{\mathbb{C}}\overline{V} < \infty \Rightarrow \overline{V}$ - simple ∎

(Proof of Thm 1)

Step 1: Consider the case $k = \mathbb{Q}(v)$, $q = v \in k$ as in the previous setup. The results of Lemmas 1–3 hold both for $V = L(\lambda)$ and $V = \tilde{L}(\lambda)$. In both cases $\overline{V}$ is the simple $\mathfrak{g}$-module of highest weight λ. As $\dim_{\mathbb{C}} \overline{V}_\mu = \dim_k V_\mu$, we see that $\dim_k L(\lambda)_\mu = \dim_k \tilde{L}(\lambda)_\mu$. As $\tilde{L}(\lambda)_\mu \twoheadrightarrow L(\lambda)_\mu$ $\forall \mu$, we see that $\tilde{L}(\lambda)_\mu \simeq L(\lambda)_\mu \Rightarrow \tilde{L}(\lambda) \simeq L(\lambda)$ and the dimensions of their weight spaces are the same as for $\mathfrak{g}$-modules.

Step 2: Assuming $\mathrm{char}(k) = 0$, q-transcendental$/\mathbb{Q}$, there is an embedding $\mathbb{Q}(v) \hookrightarrow k$ s.t. $v \mapsto q$. We shall now deduce the result from Step 1.

To distinguish, we will use subindex "k" or "$\mathbb{Q}(v)$", e.g. we have U_k ($= U_q(\mathfrak{g})$-algebra$/k$) and $U_{\mathbb{Q}(v)}$ ($= U_v(\mathfrak{g})$ - algebra $/\mathbb{Q}(v)$). Moreover, since both of them are given by the same generators and relations, we have $U_k \simeq U_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} k$. The latter also induces $U_k^{-} \simeq U^{-}_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} k$. This implies that the natural homomorphisms $M(\mu)_k \longrightarrow M(\mu)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} k$ (due to UP of Verma modules) are isomorphisms $\forall \mu$.

Recall that $\tilde{L}(\lambda)_{\mathbb{Q}(v)}$ is given by the exact sequence

$$\bigoplus_{\alpha \in \Pi} M(\lambda - (n(\alpha)+1)\alpha)_{\mathbb{Q}(v)} \xrightarrow{\sum \varphi_\alpha} M(\lambda)_{\mathbb{Q}(v)} \longrightarrow \tilde{L}(\lambda)_{\mathbb{Q}(v)} \longrightarrow 0$$

Using the above isom-s together with exactness of $\bullet \otimes_{\mathbb{Q}(v)} k$, we get

$$\bigoplus_{\alpha \in \Pi} M(\lambda - (n(\alpha)+1)\alpha)_k \xrightarrow{\sum \varphi_\alpha} M(\lambda)_k \longrightarrow \tilde{L}(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} k \longrightarrow 0$$

But there is an exactly same exact sequence with $\tilde{L}(\lambda)_k$ instead of $\tilde{L}(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} k$. Thus: $\boxed{\tilde{L}(\lambda)_k \simeq \tilde{L}(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} k}$ (1)

Theorem follows by combining (1) with the general result, which we leave as an exercise:

Exercise 2: Show that $\boxed{L(\lambda)_k \simeq L(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} k.}$ (2). ∎

(Proof of Thm 2)

It suffices to prove that any extension of two f.d. simple $U_q(\mathfrak{g})$-modules splits. In other words, given dominant $\lambda, \mu \in P$ and a S.E.S.

$$(*) \quad \boxed{0 \longrightarrow L(\lambda) \xrightarrow{\imath} V \underset{s}{\overset{\pi}{\rightleftarrows}} L(\mu) \longrightarrow 0,}$$

we want to find a "section s" s.t. $\pi s = \mathrm{Id}_{L(\mu)}$.

Let $v_\mu \in L(\mu)_\mu \setminus \{0\}$ be a highest weight vector. Since $(*)$ induces S.E.S. of weight spaces, there is $v \in V_\mu$ such that $\pi(v) = v_\mu$.

Case 1: If $E_\alpha(v) = 0 \ \forall \alpha \in \Pi$, then v is a highest wt vector of wt $= \mu$. Hence, we have $M(\mu) \twoheadrightarrow U_q(\mathfrak{g})(v)$, which factors through $\tilde{L}(\mu)$, due to [Lecture 12, Lemma 1(b)]. But $\tilde{L}(\mu) \simeq L(\mu)$ by Thm 1, hence, we get $L(\mu) \twoheadrightarrow U_q(\mathfrak{g})(v)$ which is an isomorphism and whose composition with π maps $v_\mu \mapsto v_\mu$. Thus, we obtained the "section s".

Case 2: $\exists \alpha \in \Pi$ s.t. $E_\alpha(v) \neq 0 \Rightarrow V_{\mu+\alpha} \neq 0$. As $L(\mu)_{\mu+\alpha} = 0$, this implies $L(\lambda)_{\mu+\alpha} \neq 0 \Rightarrow \boxed{\lambda > \mu}$ (strict inequality).

But now we can use the standard trick. Let us dualize $(*)$:

$$\boxed{0 \longrightarrow L(\mu)^* \longrightarrow V^* \longrightarrow L(\lambda)^* \longrightarrow 0.}$$

Arguing as in the classical case (i.e. using W-invariance of $\{\nu \mid L(\lambda)_\nu \neq 0\}$) we see that $L(\lambda)$ can be viewed as the lowest weight repr-n of the lowest weight $w_0\lambda$ (where $w_0 \in W$ is determined by $w_0 : \Pi \mapsto -\Pi$). Hence, $L(\lambda)^* \simeq L(-w_0\lambda)$. Similarly, $L(\mu)^* \simeq L(-w_0\mu)$. Therefore:

$$(**) \quad \boxed{0 \longrightarrow L(-w_0\mu) \longrightarrow V^* \longrightarrow L(-w_0\lambda) \longrightarrow 0}$$

Note that $\lambda > \mu \Rightarrow -w_0\lambda > -w_0\mu$. Hence, arguing as above, we get into Case 1, which implies $V^* \simeq L(-w_0\lambda) \oplus L(-w_0\mu)$. Dualizing once again, we get $(V^*)^* \simeq L(\lambda) \oplus L(\mu)$. It remains to use $V \simeq (V^*)^*$

Warning: We note that the natural vector space isomorphism $V \longrightarrow (V^*)^*$, $v \longmapsto (V^* \ni \varphi \mapsto \varphi(v))$ is not a morphism of $U_q(\mathfrak{g})$-modules as $S^2 \neq \mathrm{Id}$. However, the equality $S^2(u) = K_{2\rho}^{-1} \cdot u \cdot K_{2\rho}$ implies that the "twisted" map $v \longmapsto (\varphi \mapsto \varphi(K_{2\rho}^{-1} v))$ is an isom. of $U_q(\mathfrak{g})$-modules ∎

We conclude this discussion by pointing out that the idea of working with $\mathbb{Q}(v)$ and then reducing to the classical setup with the help of the homomorphism $\varphi: \mathbb{Q}[v, v^{-1}] \longrightarrow \mathbb{C}$, $v \mapsto 1$, is closely related to the way we view $U_q(\mathfrak{g})$ as a deformation of $U(\mathfrak{g})$. The following exercise generalizes [Lecture 5, Lemma 7 & Prop 3] on $\tilde{U}_q(\mathfrak{sl}_2)$.

Exercise 3: Following $\tilde{U}_q(\mathfrak{sl}_2)$, define an algebra $\tilde{U}_q(\mathfrak{g})$, generated by $\{E_\alpha, F_\alpha, K_\alpha^{\pm 1}, L_\alpha\}_{\alpha \in \Pi}$ with an explicit list of the defining relations, such that

(1) For $q^{d_\alpha} \neq 1$ $\forall \alpha$, the assignment $E_\alpha \mapsto E_\alpha$, $F_\alpha \mapsto F_\alpha$, $K_\alpha^{\pm 1} \to K_\alpha^{\pm 1}$ gives rise to an algebra isomorphism $U_q(\mathfrak{g}) \xrightarrow{\sim} \tilde{U}_q(\mathfrak{g})$.

(2) $\tilde{U}_{q=1}(\mathfrak{g}) \simeq U(\mathfrak{g})[\{K_\alpha\}_{\alpha \in \Pi}] / (K_\alpha^2 - 1)_{\alpha \in \Pi}$.