— LECTURE 14 — (03/07/2018)

Last time: Assuming $\mathrm{char}(k)=0$, q — transcendental$/\mathbb{Q}$ ("$t\mathbb{Q}$-condition" for short), we proved:
① $\hat{L}(\lambda)=L(\lambda)$ for dominant $\lambda\in Q$.
② Any f.d. $U_q(\mathfrak{g})$-module is semisimple.

Today: • Start discussion of the center $Z_q(\mathfrak{g}):=Z(U_q(\mathfrak{g}))$ by getting "upper bound"
• Define a bialgebra pairing $U_q^{\leq}\times U_q^{\geq}\xrightarrow{(\cdot,\cdot)}k$

N.B.: To prove the full description of the center, we will require some properties of $(\cdot,\cdot)$.

• Recall that $U_q(\mathfrak{g})$ is graded with $U_q(\mathfrak{g})_\nu=\{x\in U_q(\mathfrak{g})\mid K_\mu x K_\mu^{-1}=q^{(\mu,\nu)}\cdot x\ \forall \mu\in Q\}$ (this is true assuming $q\neq\sqrt{1}$).

Def: For every $\lambda\in P$, we shall use same notation to denote the homom. $U_q^0\xrightarrow{\lambda}k$ given by $\boxed{K_\mu\longmapsto q^{(\lambda,\mu)}}$

• Recall the triangular decomposition for $U_q(\mathfrak{g})$, which immediately implies

$$\boxed{U_q(\mathfrak{g})_0=U_q^0\oplus\bigoplus_{\nu>0}(U_q^-)_{-\nu}\cdot U_q^0\cdot(U_q^+)_\nu}\quad (1)$$

Def: Let $\pi: U_q(\mathfrak{g})_0\longrightarrow U_q^0$ be the projection on the 1st summand in (1).

Lemma 1: π is an algebra homomorphism.

▶ Follows from $\bigoplus_{\nu>0}(U_q^-)_{-\nu}U_q^0(U_q^+)_\nu$ being a 2-sided ideal of $U_q(\mathfrak{g})_0$ — straightforward check ■

Note that $Z_q(\mathfrak{g})\subset U_q(\mathfrak{g})_0$ (due to commuting with Cartan el-s). Hence $\pi_{|Z_q(\mathfrak{g})}: Z_q(\mathfrak{g})\to U_q^0$

Lemma 2: (a) Let $\lambda\in P$, $x\in Z_q(\mathfrak{g})$. Then x acts on $M(\lambda)$ as $\lambda(\pi(x))\cdot \mathrm{Id}_{M(\lambda)}$.
(b) Assuming "$t\mathbb{Q}$-condition" (see 1st line of this page), $\pi_{|Z_q(\mathfrak{g})}$ - injective.

▶ (a) Clearly $x(v_\lambda)=\pi(x)(v_\lambda)=\lambda(\pi(x))\cdot v_\lambda$
But x - central and $M(\lambda)$ is gen-d by v_λ $\Big\}\Rightarrow x_{|M(\lambda)}=\lambda(\pi(x))\cdot\mathrm{Id}$.

(b) Pick $x\in Z_q(\mathfrak{g})$, s.t. $\pi(x)=0$. By part (a) $\Rightarrow x_{|M(\lambda)}=0\Rightarrow x_{|L(\lambda)}=0$
Under "$t\mathbb{Q}$-condition": any f.d. module is a direct sum of $L(\lambda)$'s $\Big\}\Rightarrow$
$\Rightarrow x_{|\text{f.d. } U_q(\mathfrak{g})\text{-mod}}=0\Rightarrow x=0$. (see [Lecture 12, Thm 2]) ■

• For $\lambda\in P$, define an alg. automorphism $\gamma_\lambda: U_q^0\to U_q^0$ via $\boxed{\gamma_\lambda(K_\mu)=q^{(\lambda,\mu)}K_\mu}$
i.e. $\gamma_\lambda(x)=\lambda(x)\cdot x\ \forall x\in U_q^0$. Note that $\mu(\gamma_\lambda(x))=(\lambda+\mu)(x)$, $\gamma_\lambda\circ\gamma_\mu=\gamma_{\lambda+\mu}$.

Def: The Harish-Chandra homomorphism $HC:=\gamma_{-\rho}\circ\pi: Z_q(\mathfrak{g})\longrightarrow U_q^0$,
where as always $\rho=\sum_{\alpha\in\Pi}\varpi_\alpha=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$.

Recall the Weyl gp W, generated by the reflections $\{s_\alpha\}_{\alpha\in\Pi}$, which act on Q via $s_\alpha(\lambda)=\lambda-\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\cdot\alpha$. There is an induced action $W \curvearrowright U_q^0$. The latter is given by $w(K_\mu)=K_{w(\mu)}$.

Lemma 3: $HC(Z_q(\mathfrak{g}))\subset (U_q^0)^W$

▶ Pick $x\in Z_q(\mathfrak{g})$ and set $h:=HC(x)$. Then $x_{|M(\lambda)}=\lambda(\gamma_\rho(h))\cdot Id_{M(\lambda)}=(\lambda+\rho)(h)\cdot Id_{M(\lambda)}$.

Last two times we used that $\forall \lambda\in P,\ \alpha\in\Pi$ s.t. $(\lambda,\alpha)\geq 0$, there is a non-trivial $U_q(\mathfrak{g})$-morphism b/w Verma modules $M(\lambda-n\cdot\alpha)\longrightarrow M(\lambda)$, where $n:=1+\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}=\frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)}$ (as $(\rho,\alpha)=\frac{(\alpha,\alpha)}{2}\ \forall\alpha\in\Pi$). Hence, the constants by which x acts on these two Verma modules coincide, i.e.

$$(\lambda+\rho)(h)=(\lambda-n\alpha+\rho)(h)=(s_\alpha(\lambda+\rho))(h) \Rightarrow \boxed{(\lambda+\rho)(h)=(\lambda+\rho)(s_\alpha(h))} \quad (2)$$

The point is now to notice that (2) always holds. So far we proved it assuming $n\geq 1$. If $n=0$, then it is obvious as $s_\alpha(\lambda+\rho)=\lambda+\rho$. Finally, if $n<0$, then replace λ by $\lambda'=s_\alpha(\lambda+\rho)-\rho$ and apply above argument to λ' (note that $\{\lambda+\rho,\ s_\alpha(\lambda+\rho)\}=\{\lambda'+\rho,\ s_\alpha(\lambda'+\rho)\}$.

Thus: (2) holds $\forall\lambda\in P,\ \alpha\in\Pi \Rightarrow \boxed{\lambda(wh-h)=0\quad \forall\lambda\in P,\ w\in W} \Rightarrow wh=h\ \ \forall w\in W$. ■

Exercise 0: Explain this!

However, in contrast to $U_q(\mathfrak{sl}_2)$ studied before, $HC(Z_q(\mathfrak{g}))\neq (U_q^0)^W$. In fact, we have one more restriction on the image of $HC_{|Z_q(\mathfrak{g})}$

Def: Let $U_{ev}^0:=\bigoplus_{\mu\in(2P)\cap Q} k\cdot K_\mu$ — subspace of U_q^0

Clearly U_{ev}^0 is a subalgebra and it is W-stable

Lemma 4: $HC(Z_q(\mathfrak{g}))\subseteq (U_{ev}^0)^W$

▶ By Lemma 3: $HC(x)=\sum_{\mu\in Q}a_\mu\cdot K_\mu$ with $a_\mu=a_{w\mu}$ for any central $x\in Z_q(\mathfrak{g})$.

Recall that $\forall$ homom. $\sigma: Q\to\{\pm1\}$, we had alg. autom. $\tilde\sigma: U_q(\mathfrak{g})\circlearrowleft$ s.t. $K_\mu\mapsto\sigma(\mu)\cdot K_\mu$.

It easy to see that $\tilde\sigma$ commutes with $\pi,\gamma_\lambda \Rightarrow \tilde\sigma$ commutes with HC.

Hence: $HC(\tilde\sigma(x))=\tilde\sigma(HC(x))=\sum a_\mu\cdot\sigma(\mu)\cdot K_\mu$

But $\tilde\sigma(x)$ – central $\Rightarrow$ Lemma 3 implies that $a_\mu\cdot\sigma(\mu)=a_{w\mu}\cdot\sigma(w\mu) \Rightarrow$

$\Rightarrow a_\mu\cdot(\sigma(\mu)-\sigma(w\mu))=0$. If $a_\mu\neq 0 \Rightarrow \sigma(\mu)=\sigma(w\mu)\ \forall w \Leftrightarrow \sigma(\mu)=\sigma(s_\alpha\mu)\ \forall\alpha\in\Pi$

$\Leftrightarrow \sigma(\mu-s_\alpha(\mu))=1\ \forall\alpha\in\Pi$. But $\mu-s_\alpha(\mu)=\underbrace{\frac{2(\mu,\alpha)}{(\alpha,\alpha)}}_{n_\alpha\in\mathbb{Z}}\cdot\alpha$. Hence above requirement implies all n_α – even $\Rightarrow \mu\in 2P \Rightarrow \mu\in(2P)\cap Q$ ■

Thm 1: Assuming "tQ-condition", HC: $Z_q(\mathfrak{g}) \xrightarrow{\sim} (U^0_{ev})^W$ – isomorphism

To prove this result, we need some preparations first.

Goal (for today): Establish a pairing $U_q^{\leq} \times U_q^{\geq} \to k$.

- According to triangular decomposition $U_q^{\geq} \simeq U_q^0 \otimes U_q^+ \simeq U_q^+ \otimes U_q^0$. In particular, $\{E_I K_\mu \mid I\text{ – sequence of simple roots},\ \mu \in Q\}$ span $U_q^{\geq}$. Define the functionals $\{f_\alpha\}_{\alpha\in\Pi} \subset (U_q^{\geq})^*$:

$$\boxed{f_\alpha(E_I K_\mu) = \begin{cases} \frac{-1}{q_\alpha - q_\alpha^{-1}}, & I=(\alpha) \\ 0, & \text{otherwise}\end{cases}} \quad (3)$$

Rmk: (a) This is well-defined as there are no rel-s b/w $\{E_I K_\mu\}$ in degree α.

(b) The choice of constants may differ in some other literature.

Note that one cannot define f_I similarly to (3), since $\{E_J K_\mu\}$ are not lin. indep. Instead, for any sequence $I=(\beta_1, \ldots, \beta_r)$, we set

$$\boxed{f_I := f_{\beta_1}\cdot\ldots\cdot f_{\beta_r} \in (U_q^{\geq})^*}$$

where $(U_q^{\geq})^*$ has an alg. structure dual to coalg. str. on $U_q^{\geq}$. (Note that $f_\emptyset = \varepsilon$ – counit).

Lemma 5: (a) $f_J(E_I K_\mu) = f_J(E_I)$

(b) $f_J(E_I) = 0$ if $wt(I) \neq wt(J)$

▸ Kind of obvious. Compare to the proof of Lemma 6 below ■ ← Exercise: Work out details!

- For any $\lambda \in P$, consider the functional $k_\lambda \in (U_q^{\geq})^*$ defined via

$$\boxed{k_\lambda(E_I K_\mu) = \begin{cases} q^{-(\lambda,\mu)}, & I=\emptyset \\ 0, & \text{otherwise}\end{cases}}$$

Again, this is well-defined.

Lemma 6: (a) $\forall \lambda, \lambda' \in P$: $k_\lambda \cdot k_{\lambda'} = k_{\lambda+\lambda'}$

(b) $k_\lambda f_I = q^{-(\lambda, wt I)} \cdot f_I k_\lambda$

▸ Recall that $\Delta(E_J K_\mu) = \sum_{A,B:\, wt(A)+wt(B)=wt(J)} c^J_{A,B} \cdot E_A K_{wt(B)+\mu} \otimes E_B K_\mu$. Hence $\forall \psi \in (U_q^{\geq})^*$:

$$(k_\lambda \cdot \psi)(E_J K_\mu) = \sum_{A,B:\ldots} c^J_{A,B} \cdot k_\lambda(E_A K_{wt(B)+\mu}) \cdot \psi(E_B K_\mu) \overset{\ominus}{=} \left[\begin{array}{l}\text{But if } A\neq\emptyset \Rightarrow \text{get ZERO} \\ \text{while if } A=\emptyset \Rightarrow c^J_{\emptyset,B} = \delta_{B,J}\end{array}\right]$$
$$= k_\lambda(K_{wt(J)+\mu}) \cdot \psi(E_J K_\mu).$$

◦ If $\psi = k_{\lambda'}$, then $(k_\lambda \cdot k_{\lambda'})(E_J K_\mu) = \delta_{J,\emptyset} \cdot q^{-(\lambda,\mu)-(\lambda',\mu)-(\lambda,wt(J))} = \delta_{J,\emptyset} \cdot q^{-(\lambda+\lambda',\mu)} \Rightarrow$ (a) follows.

◦ If $\psi = f_I$, let us compute $f_I k_\lambda$:

$$f_I k_\lambda(E_J K_\mu) = \sum_{A,B} c^J_{A,B}\, f_I(E_A K_{wt(B)+\mu}) \cdot k_\lambda(E_B K_\mu) \xrightarrow[B=\emptyset,\, A=J]{\text{Only nonzero}} f_I(E_J K_\mu) \cdot k_\lambda(K_\mu)$$

The result of (b) follows as either $wt(I) \neq wt(J)$ (and both values $=0$) or $(\lambda, wt J) = (\lambda, wt I)$ ■

Recall the algebra $\overline{U}_q^{\leq} \simeq \overline{U}_q^{-} \otimes \overline{U}_q^{0}$, which has a basis $\{F_I K_\mu \mid {}^{I-\text{seq. of simple roots}}_{\mu\in Q}\}$

Consider the linear map

$$\boxed{\varphi: \overline{U}_q^{\leq} \to (U_q^{\geq})^* \text{ given by } F_I K_\mu \mapsto f_I k_\mu.}$$

<u>Lemma 7</u>: φ - alg. homom.

▸ Follows from Lemma 6 ∎

• Define a bilinear pairing

$$\boxed{(\cdot,\cdot): \overline{U}_q^{\leq} \times U_q^{\geq} \to k \quad \text{via} \quad (y,x) := \varphi(y)(x)} \qquad (F_I K_\lambda, E_J K_\mu) = f_I k_\lambda (E_J K_\mu)$$

<u>Rmk</u>: (a) For $y \in (\overline{U}_q^-)_{-\nu}$, $x \in (U_q^+)_\mu$, $\mu \neq \nu$: $\boxed{(y,x)=0}$

(b) For $y \in \overline{U}_q^-$, $x \in U_q^+$, $\lambda,\mu \in Q$: $\boxed{(yK_\lambda, xK_\mu) = q^{-(\lambda,\mu)}\cdot (y,x)}$

Let us consider the extended $(\cdot,\cdot): (\overline{U}_q^{\leq} \otimes \overline{U}_q^{\leq}) \times (U_q^{\geq} \otimes U_q^{\geq}) \to k$

$$(y_1\otimes y_2,\ x_1\otimes x_2) \longmapsto (y_1,x_1)\cdot(y_2,x_2).$$

<u>Lemma 8</u>: (a) $\forall y_1,y_2 \in \overline{U}_q^{\leq}$, $x \in U_q^{\geq}$, we have: $\boxed{(y_1 y_2, x) = (y_1\otimes y_2, \Delta(x))}$

(b) $\forall y \in \overline{U}_q^{\leq}$, $x_1,x_2 \in U_q^{\geq}$, we have: $\boxed{(y, x_1 x_2) = (\Delta(y), x_2\otimes x_1)}$

▸ (a) Immediate: $(y_1 y_2, x) = \varphi(y_1 y_2)(x) = (\varphi(y_1)\varphi(y_2))(x) = (\varphi(y_1)\otimes\varphi(y_2))(\Delta(x)) = (y_1\otimes y_2, \Delta(x))$.

(b) Following [Lecture 9], once (a) is established, it suffices to prove (b) only for y being one of the generators.

∘ $\underline{y=K_\mu} \Rightarrow \varphi(K_\mu) = k_\mu$, which is homom. $U_q^{\geq} \to k$. Then:

$(K_\mu, x_1 x_2) = k_\mu(x_1 x_2) = k_\mu(x_1)\, k_\mu(x_2) = (K_\mu \otimes K_\mu, x_2\otimes x_1) = (\Delta(K_\mu), x_2\otimes x_1)$

∘ $\underline{y=F_\alpha}$: Let us write $x_1 = E_I K_\mu$, $x_2 = E_J K_\nu$. Then:

$$(y, x_1x_2) = (F_\alpha, E_I K_\mu E_J K_\nu) = f_\alpha(E_{(I,J)}\cdot K_{\mu+\nu}\cdot q^{(\mu,\mathrm{wt}\,J)}) = \delta_{(IJ),\alpha}\cdot \frac{-q^{(\mu,\mathrm{wt}\,J)}}{q_\alpha - q_\alpha^{-1}}$$

$$(\Delta y, x_2\otimes x_1) = (F_\alpha\otimes K_\alpha^{-1} + 1\otimes F_\alpha, E_J K_\nu \otimes E_I K_\mu) = \frac{-1}{q_\alpha - q_\alpha^{-1}}\left(\delta_{I,\phi}\delta_{J,\alpha}\cdot q^{(\alpha,\mu)} + \delta_{J,\phi}\delta_{I,(\alpha)}\right)$$

Thus, we see that both terms in two lines are nonzero iff $\begin{cases} I=\phi, J=(\alpha) \\ \text{or} \\ I=(\alpha), J=\phi \end{cases}$.

Moreover, it is easy to see that coeff-s match up. ∎

<u>Lemma 9</u>: $\forall x \in U_q^{\geq}$, $\alpha \neq \beta \in \Pi$, we have $\boxed{(u_{\alpha\beta}^-, x) = 0.}$

▸ First of all $(u_{\alpha\beta}^-, x) = 0$ if $\mathrm{wt}(x) \neq r\alpha+\beta$ with $r = 1-a_{\alpha\beta}$. Remains to prove $(u_{\alpha\beta}^-, E_I) = 0$ for all I with $\mathrm{wt}(I) = r\alpha+\beta$. Write $I = (\gamma, I')$ ($\gamma = \alpha$ or β). By Lemma 8: $(u_{\alpha\beta}^-, E_I) = (\Delta(u_{\alpha\beta}^-), E_{I'}\otimes E_\gamma)$. But $\mathrm{wt}(\gamma), \mathrm{wt}(I') \neq r\alpha+\beta$, while $\Delta(u_{\alpha\beta}^-) = u_{\alpha\beta}^-\otimes K_\alpha^{-r}K_\beta^{-1} + 1\otimes u_{\alpha\beta}^-$ [Hwk 5, Prob 3]

Hence, we get ZERO $\Rightarrow (u_{\alpha\beta}^-, x) = 0\ \forall x$. ∎

As a corollary of Lemmas 8–9, we get:

Prop 1: There exists a unique bilinear pairing $(\cdot,\cdot): U_q^{\leq} \times U_q^{\geq} \to k$ s.t.

$$\boxed{\begin{gathered} (y, xx') = (\Delta(y), x' \otimes x), \quad (yy', x) = (y \otimes y', \Delta(x)) \\ \text{and} \\ (K_\mu, K_\nu) = q^{-(\mu,\nu)}, \quad (K_\mu, E_\alpha) = 0, \quad (F_\alpha, K_\mu) = 0, \quad (F_\alpha, E_\beta) = -\frac{\delta_{\alpha\beta}}{q_\alpha - q_\alpha^{-1}} \end{gathered}}$$