Last time (before Spring Break):

① Defined a Harish-Chandra homomorphism $HC := \gamma_{-\rho} \circ \pi : Z_q(\mathfrak{g}) \hookrightarrow U_q^0$ $\left(\gamma_{-\rho}: K_\mu \mapsto q^{-(\rho,\mu)} K_\mu\right)$

Proved: $\mathrm{Im}(HC) \subseteq (U_{ev}^0)^W$, where $U_{ev}^0 := \bigoplus_{\mu \in (2P) \cap Q} k \cdot K_\mu$

Thm: Assuming "TQ-conditions" ($\mathrm{char}(k)=0$, $q$-transcendental$/\mathbb{Q}$), $HC : Z_q(\mathfrak{g}) \xrightarrow{\sim} (U_{ev}^0)^W$

This was stated last time, but the proof requires us to develop some new machinery first and will be given only next time.

② Defined a homomorphism $\varphi : \overline{U}_q^{\leq} \to (U_q^{\geq})^*$ given by $\underbrace{F_I K_\mu}_{\text{(Basis of } \overline{U}_q^{\leq} \text{ by } \Delta \text{ decomp)}} \mapsto f_I k_\mu$, where $k_\lambda(E_I K_\mu) = \begin{cases} q^{-(\lambda,\mu)}, & I = \emptyset \\ 0, & \text{otherwise} \end{cases}$ and for $I = (\beta_1, \ldots, \beta_z)$ we set $f_I := f_{\beta_1} f_{\beta_2} \cdots f_{\beta_z}$ with $f_\alpha(E_I K_\mu) = \begin{cases} \frac{-1}{q_\alpha - q_\alpha^{-1}}, & I = (\alpha) \\ 0, & \text{otherwise} \end{cases}$

We verified that the induced pairing $(\,,\,) : \overline{U}_q^{\leq} \times U_q^{\geq} \to k$ $(y,x) = \varphi(y)(x)$ satisfies $(y_1 y_2, x) = (y_1 \otimes y_2, \Delta(x))$, $(y, x_1 x_2) = (\Delta(y), x_2 \otimes x_1)$ as well as $(u_{\alpha\beta}^-, x) = 0$ $\forall x \in U_q^{\geq}$. As a corollary, we deduced:

Prop: There exists a unique bilinear pairing $(\,,\,) : U_q^{\leq} \times U_q^{\geq} \to k$ s.t.

$$(y, x_1 x_2) = (\Delta(y), x_2 \otimes x_1), \quad (y_1 y_2, x) = (y_1 \otimes y_2, \Delta(x))$$
$$(K_\mu, K_\nu) = q^{-(\mu,\nu)}, \quad (K_\mu, E_\alpha) = 0, \quad (F_\alpha, K_\mu) = 0, \quad (F_\alpha, E_\beta) = -\frac{\delta_{\alpha\beta}}{q_\alpha - q_\alpha^{-1}}$$

Today: We are gonna continue our study of this pairing

Rmk: (a) For any $y \in U_q^-$, $x \in U_q^+$, $\lambda, \mu \in Q$, we have
$$(y K_\lambda, x K_\mu) = q^{-(\lambda,\mu)} \cdot (y, x)$$

(b) For any $y \in (U_q^-)_{-\nu}$, $x \in (U_q^+)_\mu$, $\nu \neq \mu$: $(y, x) \neq 0$.

Lemma 1: $(F_\alpha^n, E_\alpha^n) = (-1)^n \cdot q_\alpha^{\frac{n(n-1)}{2}} \cdot \frac{[n]_\alpha!}{(q_\alpha - q_\alpha^{-1})^n}$ $\quad \forall \alpha \in \Pi, n \in \mathbb{Z}_{\geq 0}$.

▶ We prove this by induction. Cases $n=0,1$ are clear.

$$(F_\alpha^n, E_\alpha^n) = (F_\alpha^{n-1} \otimes F_\alpha, \Delta(E_\alpha^n)) = \Big(F_\alpha^{n-1} \otimes F_\alpha, \sum_{i=0}^n q_\alpha^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_\alpha E_\alpha^{n-i} K_\alpha^i \otimes E_\alpha^i\Big) \underset{\text{reasons}}{\overset{\text{for degree}}{=}}$$
$$= (F_\alpha^{n-1} \otimes F_\alpha, q_\alpha^{n-1} \cdot [n]_\alpha \cdot E_\alpha^{n-1} K_\alpha \otimes E_\alpha) \overset{\text{Rmk(a)}}{=} (F_\alpha^{n-1}, E_\alpha^{n-1}) \cdot (F_\alpha, E_\alpha) \cdot q_\alpha^{n-1} \cdot [n]_\alpha$$
$$\overset{\text{Ind. Assump.}}{=} (-1)^{n-1} \cdot q_\alpha^{\frac{(n-1)(n-2)}{2}} \cdot \frac{[n-1]_\alpha!}{(q_\alpha - q_\alpha^{-1})^{n-1}} \cdot \frac{-1}{q_\alpha - q_\alpha^{-1}} \cdot q_\alpha^{n-1} \cdot [n]_\alpha = (-1)^n \cdot q_\alpha^{\frac{n(n-1)}{2}} \cdot \frac{[n]_\alpha!}{(q_\alpha - q_\alpha^{-1})^n}$$ ■

As illustrated by our computations in Lemma 1, we can compute the pairing by splitting off one copy of $E_\alpha$ or $F_\alpha$ and applying either of the eq-s $(y, x_1x_2) = (\Delta(y), x_2\otimes x_1)$, $(y_1y_2, x) = (y_1\otimes y_2, \Delta(x))$.

When applying the same strategy in general, the following definitions play the key role. Recall that as we saw in [Lecture 10, Lemma 6 & Corollary after]

$$\boxed{\Delta: (U_q^+)_\mu \longrightarrow \bigoplus_{0\leq\nu\leq\mu} (U_q^+)_{\mu-\nu} K_\nu \otimes (U_q^+)_\nu}$$

Also recall that $(U_q^+)_\alpha = k\cdot E_\alpha$ $\forall \alpha\in\Pi$. In particular, we can define linear maps $\tau_\alpha, \tau'_\alpha: U_q^+ \to U_q^+$ via the following f-las ($x\in (U_q^+)_\mu$ below):

$$(1)\quad \Delta(x) = x\otimes 1 + \sum_{\alpha\in\Pi} \tau_\alpha(x) K_\alpha\otimes E_\alpha + (\text{the rest}) \qquad \tau_\alpha(x), \tau'_\alpha(x)\in (U_q^+)_{\mu-\alpha}$$

$$(2)\quad \Delta(x) = K_\mu\otimes x + \sum_{\alpha\in\Pi} E_\alpha K_{\mu-\alpha}\otimes \tau'_\alpha(x) + (\text{the rest})$$

where (the rest) refers: • to terms in $(U_q^+)_{\mu-\nu} K_\nu\otimes (U_q^+)_\nu$ ($\nu\notin\Pi$, $\nu>0$) in (1)
• to terms in $(U_q^+)_{\mu-\nu} K_\nu\otimes (U_q^+)_\nu$ ($\mu-\nu\notin\Pi$, $\mu-\nu>0$) in (2).

Note: $\tau_\alpha(1) = \tau'_\alpha(1) = 0$, $\tau_\alpha(E_\beta) = \tau'_\alpha(E_\beta) = \delta_{\alpha\beta}$ $\forall \alpha,\beta\in\Pi$.

As indicated above, these linear maps play the key role in computation of $(\cdot,\cdot)$.

**Lemma 2**: (a) For $x\in (U_q^+)_\mu$, $x'\in (U_q^+)_{\mu'}$, we have:

$$\tau_\alpha(xx') = x\cdot\tau_\alpha(x') + q^{(\alpha,\mu')}\tau_\alpha(x)\cdot x', \quad \tau'_\alpha(xx') = q^{(\alpha,\mu)}\cdot x\tau'_\alpha(x') + \tau'_\alpha(x)\cdot x'.$$

(b) For $x\in (U_q^+)_\mu$, $y\in U_q^-$, we have:

$$(F_\alpha y, x) = (F_\alpha, E_\alpha)\cdot (y, \tau'_\alpha(x)), \quad (yF_\alpha, x) = (F_\alpha, E_\alpha)\cdot (y, \tau_\alpha(x))$$

(c) We have $\tau'_\alpha(x) = \sigma\tau_\alpha\sigma(x)$, where $\sigma: U_q$ Q-antiautomorphism of [Lecture 10, Lemma 2] given by $E_\alpha\mapsto E_\alpha$, $F_\alpha\mapsto F_\alpha$, $K_\alpha^{\pm1}\mapsto K_\alpha^{\mp1}$.

▶ (a) Write $\Delta(x) = x\otimes 1 + \sum_{\alpha\in\Pi}\tau_\alpha(x)K_\alpha\otimes E_\alpha + (\ldots)$, $\Delta(x') = x'\otimes 1 + \sum_{\alpha\in\Pi}\tau_\alpha(x')K_\alpha\otimes E_\alpha + (\ldots)$ to get

$$\Delta(xx') = \Delta(x)\Delta(x') = xx'\otimes 1 + \sum_{\alpha\in\Pi}(x\tau_\alpha(x')K_\alpha\otimes E_\alpha + \tau_\alpha(x)K_\alpha x'\otimes E_\alpha) + (\ldots)$$
$$= xx'\otimes 1 + \sum_{\alpha\in\Pi}(x\tau_\alpha(x') + q^{(\alpha,\mu')}\tau_\alpha(x)\cdot x')K_\alpha\otimes E_\alpha + (\ldots)$$

which implies the 1st formula of (a).

Likewise, to get the 2nd formula, one needs to multiply

$$\Delta(x) = K_\mu\otimes x + \sum_{\alpha\in\Pi} E_\alpha K_{\mu-\alpha}\otimes\tau'_\alpha(x) + (\ldots), \quad \Delta(x') = K_{\mu'}\otimes x + \sum_{\alpha\in\Pi} E_\alpha K_{\mu'-\alpha}\otimes\tau'_\alpha(x') + (\ldots)$$

— (Continuation of Proof of Lemma 2)

(b) $(F_\alpha y, x) = (F_\alpha \otimes y, \Delta(x)) = (F_\alpha \otimes y, K_\mu \otimes x + \sum_{\beta\in\Pi} E_\beta K_{\mu-\beta} \otimes \tau'_\beta(x) + (\dots)) \xlongequal[\text{Rmk}(6)]{\text{degree reasons}}$

$(F_\alpha \otimes y, E_\alpha K_{\mu-\alpha} \otimes \tau'_\alpha(x)) \xlongequal{\text{Rmk}(a)} (F_\alpha, E_\alpha)\cdot(y, \tau'_\alpha(x))$

$(yF_\alpha, x) = (y\otimes F_\alpha, \Delta(x)) = (y\otimes F_\alpha, x\otimes 1 + \sum_{\beta\in\Pi} \tau_\beta(x) K_\beta \otimes E_\beta + (\dots)) \xlongequal[\text{Rmk}(6)]{\text{degree reasons}}$

$(y\otimes F_\alpha, \tau_\alpha(x) K_\alpha \otimes E_\alpha) \xlongequal{\text{Rmk}(a)} (F_\alpha, E_\alpha)\cdot(y, \tau_\alpha(x))$

(c) The equality $\tau'_\alpha(x) = \sigma\tau_\alpha\sigma(x)$ obviously holds for $x = 1$ or $E_\beta (\beta\in\Pi)$. Hence, it suffices to prove that it holds for $x\cdot x'$ if it holds for $x$ & $x'$.

$\sigma\tau_\alpha\sigma(xx') = \sigma\tau_\alpha(\sigma(x')\cdot\sigma(x)) \overset{(a)}{=} \sigma(\sigma(x')\cdot\tau_\alpha\sigma(x) + q^{(\alpha,\mu)}\tau_\alpha\sigma(x')\cdot\sigma(x)) \xlongequal{\text{Assumption}}$

$\tau'_\alpha(x)\cdot x' + q^{(\alpha,\mu)} x\cdot\tau'_\alpha(x') \xlongequal{(a)} \tau'_\alpha(xx')$,

where $x\in(U_q^+)_\mu$, $x'\in(U_q^+)_{\mu'}$.

$\blacksquare$

This construction can be applied to the case of $U_q^-$ in the same way.

Recall: $\boxed{\Delta: (U_q^-)_{-\mu} \longrightarrow \bigoplus_{0\le\nu\le\mu} (U_q^-)_{-\nu}\otimes(U_q^-)_{-(\mu-\nu)}K_\nu^{-1}}$

Thus, we have linear maps $\tau_\alpha, \tau'_\alpha: U_q^- \circlearrowleft$, s.t. $\tau_\alpha, \tau'_\alpha: (U_q^-)_{-\mu}\to(U_q^-)_{-(\mu-\alpha)}$ are determined by (assume $y\in(U_q^-)_{-\mu}$):

$$(3)\quad \Delta(y) = y\otimes K_\mu^{-1} + \sum_{\alpha\in\Pi}\tau_\alpha(y)\otimes F_\alpha K_{\mu-\alpha}^{-1} + (\text{the rest})$$

$$(4)\quad \Delta(y) = 1\otimes y + \sum_{\alpha\in\Pi} F_\alpha\otimes\tau'_\alpha(y)K_\alpha^{-1} + (\text{the rest})$$

where (the rest) refers to terms in $(U_q^-)_{-\nu}\otimes(U_q^-)_{-(\mu-\nu)}K_\nu^{-1}$ with $\begin{cases}\mu-\nu\notin\Pi,\ \mu-\nu>0 & \text{in (3)}\\ \nu\notin\Pi,\ \nu>0 & \text{in (4)}\end{cases}$

The following is the analogue of Lemma 2:

Lemma 3: (a) For $y\in(U_q^-)_{-\mu}$, $y'\in(U_q^-)_{-\mu'}$, we have:

$\tau_\alpha(yy') = q^{(\alpha,\mu)} y\tau_\alpha(y') + \tau_\alpha(y)y'$, $\tau'_\alpha(yy') = y\tau'_\alpha(y') + q^{(\alpha,\mu')}\tau'_\alpha(y)y'$.

(b) For $y\in(U_q^-)_{-\mu}$, $x\in U_q^+$, we have:

$(y, E_\alpha x) = (F_\alpha, E_\alpha)\cdot(\tau_\alpha(y), x)$, $(y, xE_\alpha) = (F_\alpha, E_\alpha)\cdot(\tau'_\alpha(y), x)$

(c) We have $\tau'_\alpha(y) = \sigma\tau_\alpha\sigma(y)$ $\forall y\in U_q^-$.

(d) We have $\tau_\alpha(y) = \omega\tau'_\alpha\omega(y)$, $\tau'_\alpha(y) = \omega\tau_\alpha\omega(y)$ $\forall y\in U_q^-$.

where $\omega: U_q\circlearrowleft$ is the Cartan involution, given by $E_\alpha\mapsto F_\alpha$, $F_\alpha\mapsto E_\alpha$, $K_\alpha^{\pm1}\to K_\alpha^{\mp1}$

(Proof of Lemma 3)

Parts (a–c) – similar to Lemma 2.

(d) Let us prove only the first equality as the $2^{nd}$ is similar.

The equality $\tau_\alpha(y) = \omega\tau'_\alpha\omega(y)$ is clear for $y=1$ or $F_\beta$ $(\beta\in\Pi)$.

It remains to check that it holds for $yy'$ if it holds for $y$ & $y'$.

$$\omega\tau'_\alpha\omega(yy') = \omega\tau'_\alpha(\omega(y)\cdot\omega(y')) \overset{\text{Lemma 2}}{=} \omega\left(q^{(\alpha,\mu)}\cdot\omega(y)\cdot\tau'_\alpha\omega(y') + \tau'_\alpha\omega(y)\cdot\omega(y')\right) \overset{\text{Assumption}}{=\!=}$$

$$q^{(\alpha,\mu)}\cdot y\cdot\tau_\alpha(y') + \tau_\alpha(y)\cdot y' \overset{\text{Lemma 3(a)}}{=\!=} \tau_\alpha(yy'),$$

where $y\in(U_q^-)_{-\mu}$, $y'\in(U_q^-)_{-\mu'}$ ∎

Lemma 4: For all $x\in U_q^+$, $y\in U_q^-$, we have $(\omega(x),\omega(y)) = (y,x)$

▶ It suffices to prove this for homogeneous el-s, i.e. $x\in(U_q^+)_\mu$, $y\in(U_q^-)_{-\nu}$

If $\mu\neq\nu$, both sides are ZERO. Hence, assume $\nu=\mu$. If $\mu=0$ – clear.

Otherwise, let us prove by induction. It suffices to treat $y$ of the form $y=F_\alpha y'$, $y'\in(U_q^-)_{-\mu+\alpha}$.

$$(F_\alpha y', x) \overset{\text{Lemma 2}}{=} (F_\alpha, E_\alpha)\cdot(y', \tau'_\alpha(x)) \overset{\text{Induction Assumption}}{=\!=} (F_\alpha, E_\alpha)\cdot(\omega\tau'_\alpha(x), \omega(y')) \overset{\text{Lemma 3(d)}}{=\!=}$$

$$(F_\alpha, E_\alpha)\cdot(\tau_\alpha\omega(x), \omega(y')) \overset{\text{Lemma 3(b)}}{=\!=} (\omega(x), E_\alpha\omega(y')) = (\omega(x), \omega(F_\alpha y')) = (\omega(x), \omega(y))$$ ∎

Exercise 1: (a) Verify $(\sigma(y),\sigma(x)) = (y,x)$ $\forall\, x\in U_q^+, y\in U_q^-$

(b) Verify $(S(y), S(x)) = (y,x)$ $\forall\, x\in U_q^{\geq}, y\in U_q^{\leq}$

Lemma 5: For $\alpha\in\Pi$, $y\in(U_q^-)_{-\mu}$, $x\in(U_q^+)_\mu$, the following equalities hold:

$$E_\alpha y - yE_\alpha = \frac{1}{q_\alpha - q_\alpha^{-1}}\cdot\left(K_\alpha\tau_\alpha(y) - \tau'_\alpha(y)\cdot K_\alpha^{-1}\right)$$

$$F_\alpha x - xF_\alpha = \frac{-1}{q_\alpha - q_\alpha^{-1}}\left(\tau_\alpha(x)K_\alpha - K_\alpha^{-1}\tau'_\alpha(x)\right)$$

▶ The proof of the $1^{st}$ equality is by induction. Base: $y=1$ or $F_\beta$ $(\beta\in\Pi)$ – clear.

It remains to prove that it holds for $y\cdot y'$ if it holds for $y$ & $y'$. Let $y\in(U_q^-)_{-\mu}$, $y'\in(U_q^-)_{-\mu'}$.

$$[E_\alpha, yy'] = [E_\alpha, y]\cdot y' + y\cdot[E_\alpha, y'] \overset{\text{Induction Assump.}}{=\!=} \frac{1}{q_\alpha - q_\alpha^{-1}}\begin{pmatrix} K_\alpha\cdot\tau_\alpha(y)\cdot y' - \tau'_\alpha(y)\cdot K_\alpha^{-1}\cdot y' \\ + yK_\alpha\tau_\alpha(y') - y\tau'_\alpha(y')K_\alpha^{-1}\end{pmatrix}$$

$$= \frac{1}{q_\alpha - q_\alpha^{-1}}\left(K_\alpha\left(\tau_\alpha(y)\cdot y' + q^{(\alpha,\mu)}y\tau_\alpha(y')\right) - \left(q^{(\alpha,\mu')}\tau'_\alpha(y)y' + y\tau'_\alpha(y')\right)K_\alpha^{-1}\right)$$

$$\overset{\text{Lemma 3}}{=\!=} \frac{1}{q_\alpha - q_\alpha^{-1}}\left(K_\alpha\tau_\alpha(yy') - K_\alpha^{-1}\tau'_\alpha(yy')\right)$$

To deduce the $2^{nd}$ equality, apply the Cartan involution to the first ∎

**Prop 1**: Assuming "TQ-conditions", the restriction $(\ ,\ ) : (U_q^-)_{-\mu} \times (U_q^+)_\mu \to k$ is non-degenerate $\forall \mu \geq 0$.

▸ Note that $\dim (U_q^-)_{-\mu} = \dim (U_q^+)_\mu$. Hence it suffices to show that if $y \in (U_q^-)_{-\mu}$ satisfies $(y,x)=0 \ \forall x \in (U_q^+)_\mu$ (and hence $\forall x \in U_q^+$), then $y=0$.
This is obvious for $\mu=0$. For $\mu>0$ the proof is by induction.
Let us assume that $y \in (U_q^-)_{-\mu}$ is as above and the claim is proved $\forall \nu < \mu$.
Then: $(y, E_\alpha x) = 0 = (y, x E_\alpha)$ implies by Lemma 3 that $(\tau_\alpha(y), x) = 0 = (\tau'_\alpha(y), x)$ $\forall x, \forall \alpha \in \Pi$
By the induction assumption, the latter implies $\tau_\alpha(y) = 0 = \tau'_\alpha(y) \ \forall \alpha$.
Hence, $E_\alpha$ commutes with $y$ $\forall \alpha \in \Pi$, due to Lemma 5. It remains to prove:

**Claim**: If $y \in (U_q^-)_{-\mu}$ $(\mu>0)$ commutes with all $\{E_\alpha\}_{\alpha\in\Pi}$, then $y=0$.

▸ For any dominant $\lambda \in P_+$, consider the f.d. $U_q(\mathfrak{g})$-module $L(\lambda)$. Let $v_\lambda$ be its highest wt vector. Then, $E_\alpha(y v_\lambda) = y E_\alpha(v_\lambda) = 0 \Rightarrow y v_\lambda$ - h.wt. vector
But $L(\lambda)$ - f.d. simple $\Rightarrow$ no highest wt. vectors in degrees $< \lambda \Rightarrow y v_\lambda = 0$.
But, under "TQ-conditions" any f.d. $U_q(\mathfrak{g})$-module is $\simeq \oplus L(\lambda)$ (Lecture 13, Thm 2)
$\Rightarrow y$ acts trivially on all f.d. $U_q(\mathfrak{g})$-reps $\Rightarrow y=0$ [Lecture 12, Thm 2] □
This completes our proof of Prop 1 ■

**Rmk**: As a corollary of Prop 1 and Lemma 4, the bilinear form $(\ ,\ )' : U_q^+ \times U_q^+ \to k$ given by $(x,x')' := (\omega(x), x')$ is symmetric and non-degenerate.

**Lemma 6**: For any $\lambda \in Q$, $\alpha \in \Pi$, $x \in (U_q^+)_\mu$, $y \in (U_q^-)_{-\nu}$, we have

$$\mathrm{ad}(E_\alpha)(y K_\lambda x) = y K_\lambda \left(q^{-(\lambda,\alpha)} E_\alpha x - q^{(\mu-\nu,\alpha)} x E_\alpha\right) + \frac{\left(q^{-(\nu-\alpha,\alpha)} \tau_\alpha(y) K_{\lambda+\alpha} - \tau'_\alpha(y) K_{\lambda-\alpha}\right) x}{q_\alpha - q_\alpha^{-1}}$$

$$\mathrm{ad}(F_\alpha)(y K_\lambda x) = q^{-(\mu,\alpha)} \left(F_\alpha y - q^{-(\lambda,\alpha)} y F_\alpha\right) K_{\lambda+\alpha} x + \frac{y\left(q^{-(\mu-\alpha,\alpha)} K_\lambda \tau'_\alpha(x) - q^{-2(\mu-\alpha,\alpha)} K_{\lambda+2\alpha} \tau_\alpha(x)\right)}{q_\alpha - q_\alpha^{-1}}.$$

This follows by combining Lemma 5 and [Lecture 11, Lemma 1].

**Exercise 2**: Prove Lemma 6.

Next, we define the bilinear $k$-form on the entire q.gp. $U_q(\mathfrak{g})$.
Due to the triangular decomposition $m: U_q^- \otimes U_q^0 \otimes U_q^+ \xrightarrow{\sim} U_q(\mathfrak{g})$, hence,
$\bigoplus_{\mu,\nu\geq 0} ((U_q^-)_{-\nu} K_\nu) \otimes U_q^0 \otimes (U_q^+)_\mu \xrightarrow{\sim} U_q(\mathfrak{g})$. Choose a square root $q^{1/2}$.

Define $\langle \cdot, \cdot \rangle : U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \longrightarrow k(q^{1/2})$ via

$$\boxed{\langle (yK_\nu)\cdot K_\lambda \cdot x,\ (y'K_{\nu'})\cdot K_{\lambda'}\cdot x' \rangle = (y',x)(y,x')\, q^{(2\rho,\nu)-\frac{1}{2}(\lambda,\lambda')}}$$

for $x \in (U_q^+)_\mu$, $x' \in (U_q^+)_{\mu'}$, $y \in (U_q^-)_{-\nu}$, $y' \in (U_q^-)_{-\nu'}$.

The following two results will be proved next time:

Prop 2: We have $\langle ad(u)v, v' \rangle = \langle v, ad(S(u))v' \rangle$ $\forall u, v, v' \in U_q(\mathfrak{g})$.

Prop 3: Assuming "TQ-conditions", if $\langle v, v' \rangle = 0$ $\forall v \in U_q(\mathfrak{g}) \Rightarrow v' = 0$.

Exercise 3: Verify that Prop 2 is equivalent to the fact that

$$U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \longrightarrow k, \quad v \otimes v' \longmapsto \langle v, v' \rangle$$

is a $U_q(\mathfrak{g})$-morphism, where the RHS is a trivial module, while LHS is the tensor square of adjoint representation.