— LECTURE 16 — (03/28/2018)

Last time we ended up with a bilinear pairing $\langle\cdot,\cdot\rangle : U_q(\mathfrak{g})\times U_q(\mathfrak{g})\to k$ given via

$$\boxed{\langle (yK_\nu)\cdot K_\lambda\cdot x,\ (y'K_{\nu'})\cdot K_{\lambda'}\cdot x'\rangle := (y',x)\cdot(y,x')\cdot q^{(2\rho,\nu)}\cdot (q^{1/2})^{-(\lambda,\lambda')}}$$

for $x\in (U_q^+)_\mu$, $x'\in (U_q^+)_{\mu'}$, $y\in (U_q^-)_{-\nu}$, $y'\in (U_q^-)_{-\nu'}$.

Prop 1: $\langle \mathrm{ad}(u)v, v'\rangle = \langle v, \mathrm{ad}(S(u))v'\rangle \quad \forall u,v,v'\in U_q(\mathfrak{g})$.

Rmk: According to [Hwk 6, Exercise 8], this is equivalent to saying that $\langle\ ,\ \rangle$ is invariant, i.e. it gives a $U_q(\mathfrak{g})$-morphism $U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})\to k$.

▶ By linearity, we may assume $v=(yK_\nu)K_\lambda x$, $v'=(y'K_{\nu'})K_{\lambda'}x'$ as above.
It suffices to prove the claim for u being a generator, i.e. $K_\alpha, E_\alpha, F_\alpha$.

• $u=K_\alpha$

As $\mathrm{ad}(K_\alpha)(z)=K_\alpha z K_\alpha^{-1}$, $\mathrm{ad}(S(K_\alpha))(z)=K_\alpha^{-1}zK_\alpha$, we get:

$$\langle \mathrm{ad}(K_\alpha)v, v'\rangle = \langle K_\alpha v K_\alpha^{-1}, v'\rangle = \langle v,v'\rangle\cdot q^{(\alpha,\mu-\nu)}$$

$$\langle v, \mathrm{ad}(S(K_\alpha))v'\rangle = \langle v, K_\alpha^{-1}v'K_\alpha\rangle = \langle v,v'\rangle\cdot q^{-(\alpha,\mu'-\nu')}$$

But: if $\mu\neq\nu'$ or $\mu'\neq\nu$, then clearly $\langle v,v'\rangle=0$,
while if $\mu=\nu'$ & $\mu'=\nu$, then $(\alpha,\mu-\nu)=-(\alpha,\mu'-\nu')$ ⇒ still get the desired equality.

• $u=E_\alpha$

Recall $S(E_\alpha)=-K_\alpha^{-1}E_\alpha \Rightarrow \mathrm{ad}(S(E_\alpha))=-\mathrm{ad}(K_\alpha^{-1})\mathrm{ad}(E_\alpha)$.

[Lecture 15, Lemma 6]
"
[Hwk 6, Problem 7] :

$$\boxed{\begin{aligned}\mathrm{ad}(E_\alpha)(v) = {} & q^{-(\nu,\alpha)}\cdot(yK_\nu)\cdot K_\lambda\left(q^{-(\lambda,\alpha)}E_\alpha x - q^{(\mu,\alpha)}xE_\alpha\right) + \\ & + \frac{1}{q_\alpha-q_\alpha^{-1}}\cdot q^{-(\nu-\alpha,\alpha)}\left((\tau_\alpha(y)K_{\nu-\alpha})K_{\lambda+2\alpha}-(\tau'_\alpha(y)K_{\nu-\alpha})K_\lambda\right)x\end{aligned}} \quad (1)$$

By above f-la, we also get:

$$\boxed{\begin{aligned}\mathrm{ad}(S(E_\alpha))(v') = {} & -q^{-(\alpha,\alpha)}\cdot(y'K_{\nu'})K_{\lambda'}\cdot\left(q^{-(\mu'+\lambda',\alpha)}E_\alpha x' - x'E_\alpha\right) - \\ & - \frac{q^{-(\mu',\alpha)}}{q_\alpha-q_\alpha^{-1}}\cdot\left((\tau_\alpha(y')K_{\nu'-\alpha})K_{\lambda'+2\alpha} - q^{(\nu'-\alpha,\alpha)}(\tau'_\alpha(y')K_{\nu'-\alpha})K_{\lambda'}\right)x'\end{aligned}} \quad (2)$$

(Continuation of the proof of Prop 1)

As $xE_\alpha, E_\alpha x \in (\mathcal{U}_q^+)_{\mu+\alpha}$, $\tau_\alpha(y), \tau'_\alpha(y) \in (\mathcal{U}_q^-)_{-(\nu-\alpha)}$, it follows from f-la (1) that $\langle ad(E_\alpha)v, v'\rangle = 0$ unless one of the following holds:

(I) $\nu' = \mu+\alpha$, $\nu = \mu'$

(II) $\nu' = \mu$, $\nu = \mu'+\alpha$

Case (I)

$$\boxed{\langle ad(E_\alpha)v, v'\rangle = q^{-(\nu,\alpha)}\cdot(y,x')\cdot q^{(2\rho,\nu)}\cdot(q^{1/2})^{-(\lambda,\lambda')} \times \left(q^{-(\lambda,\alpha)}(y', E_\alpha x) - q^{(\mu,\alpha)}(y', xE_\alpha)\right)}$$

$$\boxed{\begin{aligned}\langle v, ad(S(E_\alpha))v'\rangle = &\frac{-1}{q_\alpha - q_\alpha^{-1}}\cdot q^{-(\mu',\alpha)}\cdot q^{(2\rho,\nu)}\cdot(y,x')\times \\ &\left(((\tau_\alpha(y'),x)\cdot(q^{1/2})^{-(\lambda,\lambda'+2\alpha)} - q^{(\nu'-\alpha,\alpha)}\cdot(\tau'_\alpha(y'),x)\cdot(q^{1/2})^{-(\lambda,\lambda')}\right)\end{aligned}}$$

As $\nu = \mu'$, $\nu' - \alpha = \mu$, the equality of these two quantities is equivalent to:

$$\boxed{q^{-(\lambda,\alpha)}\cdot(y', E_\alpha x) - q^{(\mu,\alpha)}\cdot(y', xE_\alpha) = \frac{-q^{-(\lambda,\alpha)}}{q_\alpha - q_\alpha^{-1}}\cdot(\tau_\alpha(y'),x) + \frac{q^{(\mu,\alpha)}}{q_\alpha - q_\alpha^{-1}}\cdot(\tau'_\alpha(y'),x)}$$

which follows from $(y', E_\alpha x) = -\frac{1}{q_\alpha - q_\alpha^{-1}}(\tau_\alpha(y'),x)$, $(y', xE_\alpha) = -\frac{1}{q_\alpha - q_\alpha^{-1}}(\tau'_\alpha(y'),x)$ established in [Lecture 15, Lemma 3].

Exercise 1: (a) Work out Case (II).

(b) Verify the following: $\omega\circ S\circ ad(F_\alpha) = q_\alpha^2\, ad(E_\alpha)\circ\omega\circ S$
$\omega\circ S\circ ad(S(F_\alpha)) = q_\alpha^2\, ad(-E_\alpha K_\alpha^{-1})\circ\omega\circ S$.

(c) Deduce from (b) that $\langle ad(F_\alpha)v, v'\rangle = \langle v, ad(S(F_\alpha))v'\rangle$ once we already know the claim for $u = K_\alpha$ or E_α.

This completes the proof of Prop. 1.

Prop 2: Assuming "TQ-conditions" if $\langle v,u\rangle = 0$ $\forall v \in U_q(\mathfrak{g})$, then $u=0$.

▶ Due to the "orthogonality" of different graded pieces, we can assume $u \in (U_q^-)_{-\nu}\cdot U_q^0\cdot (U_q^+)_\mu$ for some $\nu,\mu \in Q$.

- Choose a basis $\{x_i^\mu\}_{i=1}^{N_\mu}$ of $(U_q^+)_\mu$ and let $\{y_i^\mu\}_{i=1}^{N_\mu}$ be the dual basis of $(U_q^-)_{-\mu}$ w.r.t. $(\cdot,\cdot)$ which was shown to be non-degen. in [Lect 15, Prop 1].

Then $\{(y_i^\nu K_\nu)\cdot K_\lambda \cdot x_j^\mu \mid {}^{1\le i\le N_\nu,\ 1\le j\le N_\mu}_{\lambda\in Q}\}$ — basis of $(U_q^-)_{-\nu}\cdot U_q^0\cdot(U_q^+)_\mu$, s.t.

$$\langle (y_{j'}^\mu K_\mu)K_{\lambda'}x_{i'}^\nu, (y_i^\nu K_\nu)K_\lambda x_j^\mu\rangle = \delta_{i',i}\,\delta_{j',j}\cdot q^{(2\rho,\nu)}\cdot (q^{1/2})^{-(\lambda,\lambda')}$$

- Let us now decompose u in the above basis as

$$u=\sum_{i,j,\lambda} a_{ij\lambda}\cdot (y_i^\nu K_\nu)K_\lambda x_j^\mu, \quad a_{ij\lambda}\in k.$$

Then: $\langle y_j^\mu K_\mu\cdot K_{\lambda'}\cdot x_i^\nu, u\rangle = 0$ $\forall i,j,\lambda'$ $\Rightarrow$ $\boxed{\sum_\lambda a_{ij\lambda}\cdot q^{-\frac{1}{2}(\lambda,\lambda')}=0 \quad \forall i,j,\lambda'}$

If $q\ne\sqrt{1}$ $\Rightarrow$ $q^{(\cdot,-\frac12\lambda)}$ are pairwise distinct characters $Q\to k^*$ and hence are linear indep. (by Artin's thm) $\Rightarrow a_{ij\lambda}=0$ $\forall i,j,\lambda$ $\Rightarrow u=0$ ∎

Rem: We had to give some details due to U_q^0 being infinite-dim.

Prop 3: Given any bilinear map $\varphi: (U_q^-)_{-\mu}\times(U_q^+)_\nu \to k$ and $\lambda\in Q$ $\exists\, v\in (U_q^-)_{-\nu}K_{\lambda+\nu}(U_q^+)_\mu$ such that $\forall x\in(U_q^+)_\nu$, $y\in(U_q^-)_{-\mu}$, $\lambda'\in Q$:

$$\langle (yK_\mu)K_{\lambda'}x, v\rangle = \varphi(y,x)\cdot(q^{1/2})^{-(\lambda,\lambda')}$$

▶ Set $v := \sum_{i,j}\varphi(y_j^\mu, x_i^\nu)\, q^{-(2\rho,\mu)}\cdot (y_i^\nu K_\nu)K_\lambda x_j^\mu$.

By arguments in the proof of Prop 2:

$$\langle y_{j'}^\mu K_\mu K_{\lambda'} x_{i'}^\nu, v\rangle = \varphi(y_{j'}^\mu, x_{i'}^\nu)\cdot \cancel{q^{-(2\rho,\mu)}}\cdot\cancel{q^{(2\rho,\mu)}}\cdot (q^{1/2})^{-(\lambda,\lambda')}$$

∎

Now we are ready to prove finally that $HC: Z_q(\mathfrak{g})\to (U_{ev}^0)^W$ – isom. (under "TQ-conditions").

Lemma 1 (Assuming TQ-conditions): Let M be a fin. dim. $U_q(\mathfrak{g})$-module such that all weights λ of M satisfy $2\lambda \in Q$. Then, for any $m \in M$, $f \in M^*$ $\exists!\, u \in U_q(\mathfrak{g})$ s.t. the matrix coefficient $c_{f,m}(\cdot)$ equals $\langle \cdot, u\rangle$, i.e. $c_{f,m}(v) = \langle v, u\rangle$ $\forall v \in U_q(\mathfrak{g})$ (where $c_{f,m}(v) := f(v(m))$)

► By linearity, it suffices to assume m, f – homogenious el-s, i.e. $m \in M_\lambda$ and $f \in M^*_{-\lambda'}$ for some $\lambda, \lambda' \in P$ (recall that latter means $f(M_{\lambda''}) = 0$ if $\lambda'' \neq \lambda'$).
Note that $(U_q^-)_{-\mu} U_q^0 (U_q^+)_\nu\, m \subset M_{\lambda+\nu-\mu} \Rightarrow c_{f,m}|_{(U_q^-)_{-\mu} U_q^0 (U_q^+)_\nu} = 0$ unless $\lambda' = \lambda + \nu - \mu$.
We also note that $\{\nu \geq 0 \mid (U_q^+)_\nu m \neq 0\}$ – finite as M is fin.dim.
Thus, decomposing $U_q(\mathfrak{g}) = \bigoplus_{\mu,\nu} (U_q^-)_{-\mu} U_q^0 (U_q^+)_\nu$, we see that $c_{f,m}$ is nonZERO only on finitely many of these summands.

Let $x \in (U_q^+)_\nu$, $y \in (U_q^-)_{-\mu}$, $\eta \in Q$, then:

$$c_{f,m}((yK_\mu)K_\eta x) = f(yK_\mu K_\eta x(m)) = q^{(\eta, \lambda+\nu)} f(yK_\mu x m) = \underbrace{f(yK_\mu x m) \cdot (q^{1/2})^{(\eta, -2\lambda-2\nu)}}$$

But as $2\lambda \in Q$, $2\nu \in Q$, we see that Prop 3 can be applied to the RHS
$\Rightarrow \exists\, u_{\nu\mu} \in (U_q^-)_{-\nu} U_q^0 (U_q^+)_\mu$ such that $\langle v, u_{\nu\mu}\rangle = c_{f,m}(v)$ $\forall v \in (U_q^-)_{-\mu} U_q^0 (U_q^+)_\nu$.

Finally, we set $u := \sum_{\nu,\mu} u_{\nu\mu}$ with the sum only over those pairs (ν, μ) s.t. $c_{f,m}|_{(U_q^-)_{-\mu} U_q^0 (U_q^+)_\nu} \neq 0$ and as noticed above this sum is finite
! Uniqueness is due to Prop 2. ■

Lemma 2 (Assuming TQ-conditions): Let $\lambda \in P$ be a dominant weight such that $2\lambda \in Q$ (i.e. $\lambda \in P \cap \frac{1}{2}Q$). Then:
(a) $\exists!\, z_\lambda \in U_q(\mathfrak{g})$ such that $\langle u, z_\lambda\rangle = \mathrm{Tr}_{L(\lambda)}(uK_{-2\rho})$ $\forall u \in U_q(\mathfrak{g})$
(b) z_λ is central, i.e. $z_\lambda \in Z_q(\mathfrak{g})$.

► (a) This immediately follows from Lemma 1. Indeed, pick a basis $\{w_i\}$ of $L(\lambda)$ and the dual basis $\{w_i^*\}$ of $L(\lambda)^*$. Then:
$\mathrm{Tr}_{L(\lambda)}(uK_{-2\rho}) = \sum_i c_{w_i^*, w_i}(uK_{-2\rho}) = \sum_i c_{w_i^*, K_{-2\rho}(w_i)}(u)$ – sum of matrix coeff-s and hence z_λ exists by Lemma 1.
Uniqueness of z_λ is clear due to non-deg. of $\langle\, ,\rangle$.

(Continuation of the proof of Lemma 2)

(b) We claim that the linear map $U_q(\mathfrak{g}) \to k$, $u \mapsto \mathrm{Tr}_{L(\lambda)}(uK_{-2\rho})$ is actually

(*) a $U_q(\mathfrak{g})$-morphism, where the action on the LHS is via $\mathrm{ad}(\cdot)$, while on k - via the counit. One way to see that is to present this linear map as a composition of two $U_q(\mathfrak{g})$-morphisms

$$U_q(\mathfrak{g}) \longrightarrow \mathrm{End}_k(L(\lambda)) \longrightarrow k$$
$$\varphi \longmapsto \mathrm{Tr}_{L(\lambda)}(\varphi \circ K_{-2\rho})$$

Exercise 2: Work out details in the proof of (*).

So: $\varepsilon(u)\langle v, z_\lambda\rangle = \langle \mathrm{ad}(u)v, z_\lambda\rangle \;\; \forall\, u,v \in U_q(\mathfrak{g}) \overset{\text{Prop 2}}{\Longrightarrow} \mathrm{ad}(S(u))z_\lambda = \varepsilon(u)\cdot z_\lambda$

$\varepsilon(u)\langle v, z_\lambda\rangle = \langle v, \varepsilon(u) z_\lambda\rangle$, $\quad \langle \mathrm{ad}(u)v, z_\lambda\rangle \overset{\text{Prop 1}}{=} \langle v, \mathrm{ad}(S(u))z_\lambda\rangle$

But, $\varepsilon(u) = \varepsilon(S(u))$ due to [Lecture 2, Prop 1], S - antiautom. $\Rightarrow$

$$\Rightarrow \boxed{\mathrm{ad}(u)z_\lambda = \varepsilon(u) z_\lambda \quad \forall u \in U_q(\mathfrak{g})}$$

- Set $u = K_\alpha$ to get $K_\alpha z_\lambda K_\alpha^{-1} = z_\lambda \Rightarrow [K_\alpha, z_\lambda] = 0$
- Set $u = F_\alpha$ to get $\underset{\varepsilon(F_\alpha) z_\lambda = 0}{\mathrm{ad}(F_\alpha) z_\lambda} = [F_\alpha, z_\lambda] K_\alpha \Rightarrow [F_\alpha, z_\lambda] = 0$
- Set $u = E_\alpha$ to get $\underset{\varepsilon(E_\alpha) z_\lambda = 0}{\mathrm{ad}(E_\alpha) z_\lambda} = E_\alpha z_\lambda - K_\alpha z_\lambda K_\alpha^{-1} E_\alpha \overset{\text{above}}{=} [E_\alpha, z_\lambda]$

$\Rightarrow$

$\Rightarrow z_\lambda$ is central ∎

Let us write z_λ as $z_\lambda = \sum_{\mu \geq 0} z_{\lambda,\mu}$, where $z_{\lambda,\mu} \in (U_q^-)_{-\mu} \cdot U_q^0 \cdot (U_q^+)_\mu$.

$z_{\lambda,0} = \sum_\nu a_\nu \cdot K_\nu \quad (a_\nu \in k)$

Then: $\langle K_\mu, z_\lambda\rangle \overset{\text{Construction}}{=\!=} \mathrm{Tr}_{L(\lambda)}(K_{\mu - 2\rho}) = \sum_{\lambda'} \dim L(\lambda)_{\lambda'} \cdot q^{-2(\lambda',\rho)} \cdot q^{-\frac{1}{2}(-2\lambda',\mu)}$

$\langle K_\mu, z_\lambda\rangle = \langle K_\mu, z_{\lambda 0}\rangle = \langle K_\mu, \sum_\nu a_\nu K_\nu\rangle = \sum_\nu a_\nu \cdot (q^{-1/2})^{(\nu,\mu)}$

$\Rightarrow$

$$\Rightarrow \boxed{z_{\lambda,0} = \sum_{\lambda'} \dim L(\lambda)_{\lambda'} \cdot q^{-(2\lambda',\rho)} \cdot K_{-2\lambda'} = \sum_\eta \dim L(\lambda)_{-\frac{1}{2}\eta} \cdot q^{(\eta,\rho)} \cdot K_\eta}$$

$$\Rightarrow \boxed{HC(z_\lambda) = \gamma_{-\rho} \circ \pi(z_\lambda) = \gamma_{-\rho}(z_{\lambda,0}) = \sum_\eta \dim L(\lambda)_{-\frac{1}{2}\eta} \cdot K_\eta}$$

Thm 1 (Assuming "TQ-condition"): $HC: Z_q(\mathfrak{g}) \xrightarrow{\sim} (U_{ev}^{\circ})^W$ – isomorphism.

▸ • We already know that $HC: Z_q(\mathfrak{g}) \hookrightarrow (U_{ev}^{\circ})^W$.

• On the other hand, we note that $\{Av(\mu)\}_{\mu \in Q \cap 2P}$ span $(U_{ev}^{\circ})^W$, where $Av(\mu) := \sum_{\nu \in W(\mu)} K_\nu = \left(\sum_{w \in W} K_{w\mu}\right) \cdot \frac{1}{|W_\mu|}$ (W_μ – stabilizer of μ). Moreover, as $Av(\mu) = Av(w\mu)$ $\forall \mu, w$, it suffices to take μ to be one of the representatives of the corresponding W-orbit.

• Fix $\mu \in Q \cap 2P$ and set $\lambda := \frac{\mu}{2}$, so that $\lambda \in P \cap \frac{1}{2}Q$. Then, Lemma 2 applies and produces $z_\lambda \in Z_q(\mathfrak{g})$.

By the discussion in the end of p.5, we get

$$HC(z_\lambda) = \sum_\eta \dim L(\lambda)_\eta \cdot K_{-2\eta} = Av(\underbrace{-2\lambda}_{=-\mu}) + \sum_{\substack{\nu < \lambda \\ \nu\text{-dom.}}} \text{coeff} \cdot Av(-2\nu).$$

Hence, by induction $Av(-\mu) \in \operatorname{Im}(HC)$ ▨

Final Remarks (replacing "TQ-conditions" by "$q \neq \sqrt{1}$")

• In the next few lectures, we will see that actually

$$\boxed{(\ ,\ ): (U_q^-)_{-\mu} \times (U_q^+)_\mu \to k \text{ is non-deg. if } q \neq \sqrt{1}.}$$

Let us now explain how based on this result we can replace "TQ-condition" by "$q \neq \sqrt{1}$" everywhere else.

• First, we note that the argument in the above proof of Thm 1 shows that we always have $(U_{ev}^{\circ})^W \subseteq HC(Z_q(\mathfrak{g}))$, in particular, $\forall \nu \in Q \cap 2P$ $\exists z_\nu \in Z_q(\mathfrak{g})$ s.t. $HC(z_\nu) = Av(\nu) \Rightarrow \boxed{\pi(z_\nu) = \left(\sum_{w \in W} q^{(w\nu, \rho)} K_{w\nu}\right) \cdot \frac{1}{|W_\nu|}}$ (☆)

Lemma 3: Let $\lambda, \lambda' \in P$, λ-dominant. If $Z_q(\mathfrak{g})$ acts on the Verma modules $M(\lambda)$, $M(\lambda')$ by the same characters, then $\lambda' + \rho \in W(\lambda + \rho)$

▸ By (☆) above: $\sum_{w \in W} q^{(\nu, w(\lambda+\rho))} = \sum_{w \in W} q^{(\nu, w(\lambda'+\rho))}$ $\forall \nu \in Q \cap 2P$.

As $\lambda + \rho$ – strictly dominant, all $w(\lambda+\rho)$ – distinct $\xRightarrow{\text{Artin's Thm}}$ $\lambda' + \rho \in W(\lambda+\rho)$

(actually, we are using the fact that $Q \cap 2P \subset Q$ is of finite index) ▨

Lemma 4: For a dominant $\lambda \in P$, $\tilde{L}(\lambda) \simeq L(\lambda)$ if $q \neq \sqrt{1}$.

If $\tilde{L}(\lambda)$ is not simple, then it has a composition series of length ≥ 2 (recall $\tilde{L}(\lambda) \twoheadrightarrow L(\lambda)$). As $\dim \tilde{L}(\lambda)_\lambda = \dim L(\lambda)_\lambda = 1$, we see that $\tilde{L}(\lambda)$ has a subquotient $\simeq L(\lambda')$ for $0 \leq \lambda' < \lambda$.

But $Z_q(\mathfrak{g})$ acts by the same character on all simple subquotients of $\tilde{L}(\lambda)$ (as all of them are subquotients of Verma).

$$\left.\begin{array}{l}\text{Hence, by Lemma 3, } \lambda' + \rho \in W(\lambda+\rho). \\ \text{But } \lambda, \lambda' \text{ - dominant} \Rightarrow \lambda+\rho, \lambda'+\rho \text{ - strictly dominant}\end{array}\right\} \Rightarrow \lambda'+\rho = \lambda+\rho \Rightarrow \lambda' = \lambda \Rightarrow \text{↯}$$

■

Having established the isomorphism $\tilde{L}(\lambda) \simeq L(\lambda)$ for $q \neq \sqrt{1}$, we see that in all the remaining spots, we can replace the "TQ-conditions" by "$q \neq \sqrt{1}$".

Remark: Prove $(\,,\,): (U_q^-)_{-\mu} \times (U_q^+)_\mu \to k$ is non-deg for $q \neq \sqrt{1}$.