— LECTURE 17 — (04/02/2018)

Goal 1 for today: Construct a functorial isomorphism $M_2 \otimes M_1 \xrightarrow{\sim} M_1 \otimes M_2$ of $U_q(\mathfrak{g})$-mod-s for M_1, M_2 - fin. dimensional, generalizing our construction for $\mathfrak{g} = \mathfrak{sl}_2$ from Lecture 8.

- For every $\mu \in Q$, $\mu \geq 0$ (denoted $\mu \in Q_+$), choose an arbitrary basis $\{x_i^\mu\}_{i=1}^{N_\mu}$ of $(U_q^+)_\mu$ and let $\{y_i^\mu\}_{i=1}^{N_\mu}$ be the dual basis of $(U_q^-)_{-\mu}$ w.r.t. the non-degenerate pairing $(\,,\,): (U_q^-)_{-\mu} \times (U_q^+)_\mu \to k$ from Lecture 15, i.e. $(y_i^\mu, x_j^\mu) = \delta_{ij}$. Define

$$\boxed{\Theta_\mu := \sum_{i=1}^{N_\mu} y_i^\mu \otimes x_i^\mu \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})}$$

Clearly, Θ_μ is independent of our choice of the basis $\{x_i^\mu\}$.

Lemma 1: Recall the Cartan involution ω and the antiautomorphism σ of $U_q(\mathfrak{g})$.
(a) $(\sigma \otimes \sigma)(\Theta_\mu) = \Theta_\mu$
(b) $(\omega \otimes \omega)(\Theta_\mu) = \tau(\Theta_\mu) := \Theta_\mu^{op}$

▶ (a) $\{\sigma(x_i^\mu)\}_{i=1}^{N_\mu}$ — another basis of $(U_q^+)_\mu$ and $\{\sigma(y_i^\mu)\}_{i=1}^{N_\mu}$ is the dual (to it) basis, due to the equality $(\sigma(y), \sigma(x)) = (y, x)$ from [Hwk 6, Problem 6(a)].
(b) $\{\omega(x_i^\mu)\}_{i=1}^{N_\mu}$ - a basis of $(U_q^-)_{-\mu}$ and $\{\omega(y_i^\mu)\}_{i=1}^{N_\mu}$ is the dual basis of $(U_q^+)_\mu$, due to the equality $(\omega(x), \omega(y)) = (y, x)$ from [Lecture 15, Lemma 4]. ■

Thus, we have a family of elements $\{\Theta_\mu \mid \mu \in Q_+\}$. We set $\Theta_\mu = 0$ if $\mu \not\geq 0$.
Note that $(1,1) = 1$ and $(F_\alpha^n, E_\alpha^n) = (-1)^n \cdot q_\alpha^{\frac{n(n-1)}{2}} \cdot \frac{[n]_\alpha!}{(q_\alpha - q_\alpha^{-1})^n}$, so that

$$\boxed{\Theta_{n\alpha} = (-1)^n q_\alpha^{-\frac{n(n-1)}{2}} \cdot \frac{(q_\alpha - q_\alpha^{-1})^n}{[n]_\alpha!} = \Theta_n \leftarrow \text{from Lecture 8 for } \mathfrak{g} = \mathfrak{sl}_2}$$

Lemma 2: For any $\mu \in Q_+$, $\alpha \in \Pi$, we have:
(a) $(E_\alpha \otimes 1)\Theta_\mu + (K_\alpha \otimes E_\alpha)\Theta_{\mu-\alpha} = \Theta_\mu (E_\alpha \otimes 1) + \Theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha)$
(b) $(1 \otimes F_\alpha)\Theta_\mu + (F_\alpha \otimes K_\alpha^{-1})\Theta_{\mu-\alpha} = \Theta_\mu (1 \otimes F_\alpha) + \Theta_{\mu-\alpha}(F_\alpha \otimes K_\alpha)$
(c) $(K_\alpha \otimes K_\alpha)\Theta_\mu = \Theta_\mu (K_\alpha \otimes K_\alpha)$

Note that this is a direct generalization of [Lecture 8, Lemma 3].

(Proof of Lemma 2)

(c) Obvious as Θ_μ is of degree 0.

(a) $(E_\alpha\otimes 1)\Theta_\mu - \Theta_\mu(E_\alpha\otimes 1) = \sum_i (E_\alpha y_i^\mu - y_i^\mu E_\alpha)\otimes x_i^\mu \xlongequal[\text{Lemma 5}]{\text{Lecture 15}}$

$$= \sum_i \frac{1}{q_\alpha - q_\alpha^{-1}}\left(K_\alpha r_\alpha(y_i^\mu) - r'_\alpha(y_i^\mu)K_\alpha^{-1}\right)\otimes x_i^\mu \ominus$$

To proceed we note that $\forall x\in (U_q^+)_\nu,\ y\in (U_q^-)_{-\nu}$, we have $x=\sum_i (y_i^\nu, x)\cdot x_i^\nu,\ y=\sum_i (y, x_i^\nu) y_i^\nu$

$$\ominus \frac{1}{q_\alpha-q_\alpha^{-1}}\sum_i\left(K_\alpha\sum_j (r_\alpha(y_i^\mu), x_j^{\mu-\alpha})y_j^{\mu-\alpha} - \sum_j (r'_\alpha(y_i^\mu), x_j^{\mu-\alpha})y_j^{\mu-\alpha}K_\alpha^{-1}\right)\otimes x_i^\mu \xlongequal[\text{Lemma 3}]{\text{Lecture 15}}$$

$$= \sum_i\left(-K_\alpha\sum_j (y_i^\mu, E_\alpha x_j^{\mu-\alpha})y_j^{\mu-\alpha} + \sum_j (y_i^\mu, x_j^{\mu-\alpha}E_\alpha)y_j^{\mu-\alpha}K_\alpha^{-1}\right)\otimes x_i^\mu =$$

$$= \sum_j\left(y_j^{\mu-\alpha}K_\alpha^{-1}\otimes\sum_i (y_i^\mu, x_j^{\mu-\alpha}E_\alpha)x_i^\mu - K_\alpha y_j^{\mu-\alpha}\otimes\sum_i (y_i^\mu, E_\alpha x_j^{\mu-\alpha})x_i^\mu\right) =$$

$$= \sum_j\left(y_j^{\mu-\alpha}K_\alpha^{-1}\otimes x_j^{\mu-\alpha}E_\alpha - K_\alpha y_j^{\mu-\alpha}\otimes E_\alpha x_j^{\mu-\alpha}\right) = \Theta_{\mu-\alpha}(K_\alpha^{-1}\otimes E_\alpha) - (K_\alpha\otimes E_\alpha)\Theta_{\mu-\alpha}$$

This proves (a).

(b) Analogous!

Exercise 1: Work out details in part (b).

Let M_1, M_2 be finite-dimensional $U_q(\mathfrak{g})$-modules (of type 1). For every $\mu\in Q$, $\lambda_1,\lambda_2\in P$, we clearly have $\Theta_\mu((M_1)_{\lambda_1}\otimes(M_2)_{\lambda_2})\subset (M_1)_{\lambda_1-\mu}\otimes(M_2)_{\lambda_2+\mu}$. As M_1, M_2 – fin.dim, this implies that all, but a finitely many Θ_μ, act by ZERO on $M_1\otimes M_2$. Set

$$\boxed{\Theta = \Theta_{M_1,M_2} = \sum_{\mu\in Q_+}\Theta_\mu : M_1\otimes M_2 \longrightarrow M_1\otimes M_2}$$

Rmks: (1) In a suitable basis (just take a tensor product of properly ordered weight bases) Θ acts in an upper-triangular way with 1's on diagonal. Hence, Θ_{M_1,M_2} is unipotent and invertible.

(2) Lemma 2 can be rephrased as follows

$$\boxed{\Delta(u)\circ\Theta_{M_1,M_2} = \Theta_{M_1,M_2}\circ\bar\Delta(u)\quad \forall u\in U_q(\mathfrak{g})}$$

where $\bar\Delta$ is the twisted comultiplication s.t.

$\bar\Delta:\ K_\alpha\mapsto K_\alpha\otimes K_\alpha,\ E_\alpha\mapsto E_\alpha\otimes 1 + K_\alpha^{-1}\otimes E_\alpha,\ F_\alpha\mapsto 1\otimes F_\alpha + F_\alpha\otimes K_\alpha.$

Note that $\bar\Delta = (\sigma\otimes\sigma)\circ\Delta\circ\sigma$

Next, we pick a map

$$\boxed{\begin{array}{c} f: P\times P \longrightarrow k^* \quad \text{s.t.} \quad f(\lambda+\nu,\mu) = q^{-(\nu,\mu)} f(\lambda,\mu),\ f(\lambda,\mu+\nu) = q^{-(\nu,\lambda)} f(\lambda,\mu) \\ \text{for any } \lambda,\mu\in P,\ \nu\in Q. \end{array}}$$

For fin.dim. modules M_1, M_2, we consider the corresponding linear map

$$\boxed{\mathcal{F}: M_1\otimes M_2 \longrightarrow M_1\otimes M_2 \ \text{ given by } \ \mathcal{F}(m_1\otimes m_2) = f(\lambda,\mu)\cdot m_1\otimes m_2 \quad \forall m_1\in (M_1)_\lambda, m_2\in (M_2)_\mu}$$

<u>Theorem 1</u>: For any finite-dimensional $U_q(\mathfrak{g})$-modules M_1, M_2, the map

$$\Theta\circ\mathcal{F}\circ\tau: M_2\otimes M_1 \longrightarrow M_1\otimes M_2$$

is an isomorphism of $U_q(\mathfrak{g})$-modules.

The proof is completely analogous to our proof of [Lecture 8, Thm 2], where a particular case of $\mathfrak{g}=\mathfrak{sl}_2$ was treated. The key point is that

$$\Theta\bar{\Delta}(u) = \Delta(u)\Theta,\quad \bar{\Delta}(u)\mathcal{F} = \mathcal{F}\Delta^{op}(u) \Rightarrow \boxed{\Delta(u)(\Theta\circ\mathcal{F}) = (\Theta\circ\mathcal{F})\Delta^{op}(u).\ \forall u\in U_q(\mathfrak{g})}$$

<u>Exercise 2</u>: Work out details of the proof.

Given three fin. dim. $U_q(\mathfrak{g})$-representations M_1, M_2, M_3, we obtain three automorphisms $\Theta^f_{12}, \Theta^f_{13}, \Theta^f_{23} : M_1\otimes M_2\otimes M_3 \circlearrowleft$, e.g. $\Theta^f_{12} = (\Theta_{M_1,M_2}\mathcal{F})\otimes \mathrm{Id}$.

<u>Theorem 2</u>: We have

$$\boxed{\Theta^f_{12}\circ\Theta^f_{13}\circ\Theta^f_{23} = \Theta^f_{23}\circ\Theta^f_{13}\circ\Theta^f_{12}}$$

This is a generalization of [Lecture 8, Theorem 3].
The proof is similar to the one for $\mathfrak{g}=\mathfrak{sl}_2$, but is a bit more technical and is crucially based on the following result.

<u>Lemma 3</u>: For any $\mu\in Q_+$, we have:

(a) $$(\Delta\otimes 1)\Theta_\mu = \sum_{0\leq\nu\leq\mu} (\Theta_{\mu-\nu})_{23}(1\otimes K_\nu^{-1}\otimes 1)(\Theta_\nu)_{13}$$

(b) $$(1\otimes\Delta)\Theta_\mu = \sum_{0\leq\nu\leq\mu} (\Theta_{\mu-\nu})_{12}(1\otimes K_\nu\otimes 1)(\Theta_\nu)_{13}$$

Recall that in case $\mathfrak{g}=\mathfrak{sl}_2$, we had these equalities in the following form:

$(\Delta\otimes 1)\Theta_n = \sum_{i=0}^n (1\otimes\Theta_{n-i})\Theta''_i$, where $\Theta''_i = a_i F^i\otimes K^{-i}\otimes E^i$

$(1\otimes\Delta)\Theta_n = \sum_{i=0}^n (\Theta_{n-i}\otimes 1)\Theta'_i$, where $\Theta'_i = a_i F^i\otimes K^i\otimes E^i$

(Proof of Lemma 3)

We start from the following f-las for $x\in(U_q^+)_\mu$, $y\in(U_q^-)_{-\mu}$:

$$\boxed{\begin{aligned}
&(1)\quad \Delta(x)=\sum_{0\le\nu\le\mu}\sum_{i,j}(y_i^{\mu-\nu}y_j^\nu, x)\, x_i^{\mu-\nu}K_\nu\otimes x_j^\nu\\
&(2)\quad \Delta(y)=\sum_{0\le\nu\le\mu}\sum_{i,j}(y, x_i^{\mu-\nu}x_j^\nu)\, y_j^\nu\otimes y_i^{\mu-\nu}K_\nu^{-1}
\end{aligned}}$$

To prove (1), note that we know apriori that $\Delta(x)=\sum_{0\le\nu\le\mu}^{i,j}c_{ij}^\nu\cdot x_i^{\mu-\nu}K_\nu\otimes x_j^\nu$ for some constants $\{c_{ij}^\nu\}$, which can be found via $c_{ij}^\nu=(y_i^{\mu-\nu}\otimes y_j^\nu,\Delta(x))\underset{\text{we use the key property of }(\,,\,)}{=}(y_i^{\mu-\nu}y_j^\nu, x)$

Hence, formula (1) follows. Formula (2) is proved similarly.

Let us now prove Lemma 3(a), while part (b) is similar

$$(\Delta\otimes1)\Theta_\mu=\sum_r\Delta(y_r^\mu)\otimes x_r^\mu\overset{(2)}{=}\sum_{0\le\nu\le\mu}^{r,i,j}(y_r^\mu, x_i^{\mu-\nu}x_j^\nu)\,y_j^\nu\otimes y_i^{\mu-\nu}K_\nu^{-1}\otimes x_r^\mu=$$

$$=\sum_{0\le\nu\le\mu}^{i,j}y_j^\nu\otimes y_i^{\mu-\nu}K_\nu^{-1}\otimes\Big(\sum_r(y_r^\mu, x_i^{\mu-\nu}x_j^\nu)x_r^\mu\Big)=\sum_{0\le\nu\le\mu}^{i,j}y_j^\nu\otimes y_i^{\mu-\nu}K_\nu^{-1}\otimes x_i^{\mu-\nu}x_j^\nu=$$

$$=\sum_{0\le\nu\le\mu}(\Theta_{\mu-\nu})_{23}(1\otimes K_\nu^{-1}\otimes1)(\Theta_\nu)_{13}$$

$\square$

Exercise 3: Prove part (b) of Lemma 3.

Lemma 4: For the twisted coproduct $\tilde\Delta$ (see p.2), the following holds:

$$(\tilde\Delta\otimes1)\Theta_\mu=\sum_{0\le\nu\le\mu}(\Theta_\nu)_{13}(1\otimes K_\nu\otimes1)(\Theta_{\mu-\nu})_{23}$$

$$(1\otimes\tilde\Delta)\Theta_\mu=\sum_{0\le\nu\le\mu}(\Theta_\nu)_{13}(1\otimes K_\nu^{-1}\otimes1)(\Theta_{\mu-\nu})_{12}$$

Recall that $\tilde\Delta=(\sigma\otimes\sigma)\circ\Delta\circ\sigma$. Hence:

$$(\tilde\Delta\otimes1)\Theta_\mu=(\sigma\otimes\sigma\otimes1)(\Delta\otimes1)(\sigma\otimes1)\Theta_\mu\overset{\text{As }\sigma^2=1}{=}(\sigma\otimes\sigma\otimes\sigma)(\Delta\otimes1)(\sigma\otimes\sigma)\Theta_\mu\overset{\text{Lemma 1}}{=}$$

$$=(\sigma\otimes\sigma\otimes\sigma)(\Delta\otimes1)\Theta_\mu\overset{\text{Lemma 3}}{=}(\sigma\otimes\sigma\otimes\sigma)\Big(\sum_{0\le\nu\le\mu}(\Theta_{\mu-\nu})_{23}(1\otimes K_\nu^{-1}\otimes1)(\Theta_\nu)_{13}\Big)\underset{\sigma\text{-antiaut}}{\overset{\text{Lemma 1}}{=}}$$

$$=\sum_{0\le\nu\le\mu}(\Theta_\nu)_{13}\cdot(1\otimes K_\nu\otimes1)(\Theta_{\mu-\nu})_{23}$$

The second equality is proved completely analogously $\square$

Now we are ready to prove Theorem 2.

(Proof of Thm 2)

$$LHS = \Theta_{12} \circ \tilde{f}_{12} \circ \Theta_{13} \circ \tilde{f}_{13} \circ \Theta_{23} \circ \tilde{f}_{23}$$

Lemma 5: For any $\eta \in Q$, we have:

(a) $\tilde{f}_{12} \circ (\Theta_\eta)_{13} = (\Theta_\eta)_{13} \circ (1 \otimes K_\eta \otimes 1) \circ \tilde{f}_{12}$

(b) $\tilde{f}_{12} \circ \tilde{f}_{13} \circ (\Theta_\eta)_{23} = (\Theta_\eta)_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13}$

▻ Pick $w = m_1 \otimes m_2 \otimes m_3 \in (M_1)_\lambda \otimes (M_2)_\mu \otimes (M_3)_\nu$.

(a) $\tilde{f}_{12} \circ (\Theta_\eta)_{13}(w) = \tilde{f}_{12}\left(\sum_i y_i^\eta m_1 \otimes m_2 \otimes x_i^\eta m_3\right) = \sum_i f(\lambda - \eta, \mu) \cdot y_i^\eta m_1 \otimes m_2 \otimes x_i^\eta m_3$

$= q^{(\eta,\mu)} \sum_i f(\lambda,\mu) \cdot y_i^\eta m_1 \otimes m_2 \otimes x_i^\eta m_3 = (\Theta_\eta)_{13} \circ (1 \otimes K_\eta \otimes 1) \circ \tilde{f}_{12}(w)$.

(b) $\tilde{f}_{12} \circ \tilde{f}_{13} \circ (\Theta_\eta)_{23}(w) = \tilde{f}_{12} \circ \tilde{f}_{13}\left(\sum_i m_1 \otimes y_i^\eta m_2 \otimes x_i^\eta m_3\right) =$

$= \sum_i f(\lambda, \mu - \eta) \cdot f(\lambda, \nu + \eta) \cdot m_1 \otimes y_i^\eta m_2 \otimes x_i^\eta m_3 = f(\lambda,\mu) f(\lambda,\nu) \sum_i m_1 \otimes y_i^\eta m_2 \otimes x_i^\eta m_3$

$= (\Theta_\eta)_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13}(w)$ □

So: $\boxed{LHS = \Theta_{12} \circ \left(\sum_\eta (\Theta_\eta)_{13}(1 \otimes K_\eta \otimes 1)\right) \circ \Theta_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13} \circ \tilde{f}_{23}}$

In the same way, we can immediately derive an analogous f-la for RHS:

$\boxed{RHS = \Theta_{23} \circ \left(\sum_\eta (\Theta_\eta)_{13}(1 \otimes K_\eta^{-1} \otimes 1)\right) \circ \Theta_{12} \circ \tilde{f}_{23} \circ \tilde{f}_{13} \circ \tilde{f}_{12}}$

Obviously, $\tilde{f}_{12} \tilde{f}_{13} \tilde{f}_{23} = \tilde{f}_{23} \tilde{f}_{13} \tilde{f}_{12}$.

But, due to Lemma 4(a) & Lemma 3(a), we have:

$\Theta_{12} \circ \left(\sum_\eta (\Theta_\eta)_{13}(1 \otimes K_\eta \otimes 1)\right) \circ \Theta_{23} = \Theta_{12} \circ \left((\bar{\Delta} \otimes 1)\Theta\right)$

$\Theta_{23} \circ \left(\sum_\eta (\Theta_\eta)_{13}(1 \otimes K_\eta^{-1} \otimes 1)\right) \circ \Theta_{12} = \left((\Delta \otimes 1)\Theta\right) \circ \Theta_{12}$

due to Lemma 2, see also Rmk (2).

Thus, LHS = RHS ∎

Rmk: If we set $M_1 = M_2 = M_3 = V$, then Θ^f – solution of the quantum Yang–Baxter eq-n, cf. [Lecture 8, Corollary after Thm 3].

But it is more common to consider $\Theta^f \circ \tau$ in the context of this Rmk. (τ – permutation)

Set $R_{12} := \Theta^f_{12} \circ \tau_{12}$, $R_{23} := \Theta^f_{23} \circ \tau_{23}$, which are $U_q(\mathfrak{g})$-automorphisms of $V \otimes V \otimes V$.

Note: $R_{12} \circ R_{23} \circ R_{12} = \Theta^f_{12} \circ \tau_{12} \circ \Theta^f_{23} \circ \tau_{23} \circ \Theta^f_{12} \circ \tau_{12} = \Theta^f_{12} \circ \Theta^f_{13} \circ \tau_{12} \circ \tau_{23} \circ \Theta^f_{12} \circ \tau_{12} =$

$= \Theta^f_{12} \circ \Theta^f_{13} \circ \Theta^f_{23} \circ \tau_{12} \circ \tau_{23} \circ \tau_{12}$

$R_{23} \circ R_{12} \circ R_{23} = \Theta^f_{23} \circ \tau_{23} \circ \Theta^f_{12} \circ \tau_{12} \circ \Theta^f_{23} \circ \tau_{23} = \Theta^f_{23} \circ \Theta^f_{13} \circ \tau_{23} \circ \tau_{12} \circ \Theta^f_{23} \circ \tau_{23} =$

$= \Theta^f_{23} \circ \Theta^f_{13} \circ \Theta^f_{12} \circ \tau_{23} \circ \tau_{12} \circ \tau_{23}$

Thus, Theorem 2 implies (actually is equivalent) in this setup:

$$\boxed{R_{12}\circ R_{23}\circ R_{12} = R_{23}\circ R_{12}\circ R_{23}} \quad (*)$$

More generally, for any $r\geq 3$, consider $V^{\otimes r}$. For $1\leq i\leq r-1$, define $R_i := \Theta^f_{i,i+1}\circ \tau_{i,i+1} : V^{\otimes r} \circlearrowleft$ which is a $U_q(\mathfrak{g})$-endomorphism. According to $(*)$, $R_i\circ R_{i+1}\circ R_i = R_{i+1}\circ R_i\circ R_{i+1}$ $\forall\, 1\leq i\leq r-1$. But we also have $R_i\circ R_j = R_j\circ R_i$ if $|j-i|\geq 2$.

Conclusion: The endomorphisms $\{R_i\}_{i=1}^{r-1}$ satisfy

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} \quad \text{and } R_i R_j = R_j R_i \ (\forall\, |i-j|\geq 2).$$

Therefore, they define a representation of the braid group in $V^{\otimes r}$.

Exercise 4: Recall our explicit f-la for Θ^f in case of $\mathfrak{g}=\mathfrak{sl}_2$, see [Lecture 9, Rmk 2].

$$\Theta^f = \begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & q^{-1}-q & 1 & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix} \Rightarrow R = \Theta^f\circ\tau = \begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & q^{-1}-q & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}$$

Verify the equality $R^2=(q^{-1}-q)R+1$, or equivalently, $(qR^{-1})^2=(q^2-1)(qR^{-1})+q^2$.
Deduce that the operators $R_i' := qR_i^{-1}$ $(1\leq i\leq r-1)\in \mathrm{End}_{U_q(\mathfrak{g})}(V^{\otimes r})$ satisfy

$$\boxed{R_i' R_{i+1}' R_i' = R_{i+1}' R_i' R_{i+1}',\ R_i' R_j' = R_j' R_i' \ (j\neq i, i\pm 1),\ (R_i')^2 = (q^2-1)R_i' + q^2}$$

In other words, $\{R_i'\}$ define a representation of type A_{r-1} Hecke algebra on $V^{\otimes r}$.

Theorem 3: Let M_1, M_2, M_3 be fin. dim. $U_q(\mathfrak{g})$-representations. Assume that f also satisfies $\boxed{f(\lambda,\mu+\nu)=f(\lambda,\mu)f(\lambda,\nu),\ f(\lambda+\mu,\nu)=f(\lambda,\nu)f(\mu,\nu)\ \forall \lambda,\mu,\nu\in P}$

Then, $\Theta^f\circ\tau$ also satisfy the hexagon identities, i.e.

$$\begin{tikzcd}
 & M_1\otimes(M_3\otimes M_2) \arrow[r, "can"] & (M_1\otimes M_3)\otimes M_2 \arrow[dr, "R_{12}"] & \\
M_1\otimes(M_2\otimes M_3) \arrow[ur, "R_{23}"] \arrow[dr, "can"'] & & & (M_3\otimes M_1)\otimes M_2 \\
 & (M_1\otimes M_2)\otimes M_3 \arrow[r, "R_{12,3}"] & M_3\otimes(M_1\otimes M_2) \arrow[ur, "can"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (M_2\otimes M_1)\otimes M_3 \arrow[r, "can"] & M_2\otimes(M_1\otimes M_3) \arrow[dr, "R_{23}"] & \\
(M_1\otimes M_2)\otimes M_3 \arrow[ur, "R_{12}"] \arrow[dr, "can"'] & & & M_2\otimes(M_3\otimes M_1) \\
 & M_1\otimes(M_2\otimes M_3) \arrow[r, "R_{1,23}"] & (M_2\otimes M_3)\otimes M_1 \arrow[ur, "can"'] &
\end{tikzcd}$$

are commutative.

The proof is exactly the same as in [Lecture 9, Prop 2].

In view of Theorem 3 and Theorem 1, it is desirable to provide an example of $f : P\times P \to k^*$ satisfying the 4 relations

$$\boxed{\begin{gathered} f(\lambda+\nu,\mu)=q^{-(\nu,\mu)}f(\lambda,\mu),\ f(\lambda,\mu+\nu)=q^{-(\nu,\lambda)}f(\lambda,\mu) \quad \forall \lambda,\mu\in P,\ \nu\in Q \\ \text{and} \\ f(\lambda,\mu+\nu)=f(\lambda,\mu)f(\lambda,\nu),\ f(\lambda+\mu,\nu)=f(\lambda,\nu)f(\mu,\nu) \quad \forall\lambda,\mu,\nu\in P \end{gathered}}$$

Exercise 5: Find all such f assuming k contains certain roots of q.
find explicit values

Goal 2 for today: Algebra $k_q[G]$ (here G is the simply connected, connected, semisimple alg. gp with Lie algebra $\mathfrak{g}$)

Def: Let $k_q[G]$ be the subspace of $U_q(\mathfrak{g})^*$ spanned by matrix coefficients $\{C_{f,m} \mid m\in M$ - fin.dim. $U_q(\mathfrak{g})$-module, $f\in M^*\}$.

Lemma 6: $k_q[G]$ is a subalgebra with unit.

Let M_1, M_2 be fin.dim. $U_q(\mathfrak{g})$-representations, $m_1\in M_1$, $m_2\in M_2$, $f_1\in M_1^*$, $f_2\in M_2^*$. Then, we actually have

$$\boxed{C_{f_1,m_1}\cdot C_{f_2,m_2} = C_{f_1\otimes f_2,\, m_1\otimes m_2}}$$

To prove this, pick $u\in U_q(\mathfrak{g})$ and write $\Delta(u)=\sum_{(u)} u'\otimes u''$. Then:

$$C_{f_1\otimes f_2, m_1\otimes m_2}(u) = (f_1\otimes f_2)(u(m_1\otimes m_2)) = (f_1\otimes f_2)\Big(\sum_{(u)}(u'm_1)\otimes(u''m_2)\Big)=$$
$$=\sum_{(u)} f_1(u'm_1)f_2(u''m_2) = \sum_{(u)} C_{f_1,m_1}(u')C_{f_2,m_2}(u'') = C_{f_1,m_1}C_{f_2,m_2}(u)$$

Thus, $k_q[G]$ is closed under multiplication.

Take a trivial module $M=k$, $m=1\in M$, $f=1^*\in M^*$, then $C_{f,m}=\varepsilon$ (= counit of $U_q(\mathfrak{g})$) is clearly a unit of $k_q[G]$ ∎

Exercise 6: Prove that $k_q[G]$ naturally has a Hopf alg. structure with

(a) comultiplication determined via $\Delta^*(C_{f,m})=\sum_{i=1}^{n} C_{f,m_i}\otimes C_{f_i,m} \in k_q[G]\otimes k_q[G]$, where $\{m_i\}_{i=1}^n$ - a basis of M, $\{f_i\}_{i=1}^n$ - the dual basis of M^*

(b) counit $\varepsilon^*: k_q[G]\to k$ determined via $\varepsilon^*(C_{f,m})=f(m)$, i.e. $\varepsilon^*(\varphi)=\varphi(1)$ $\forall\varphi\in k_q[G]$

(c) antipode $S^*: k_q[G] \circlearrowleft$ determined via $S^*(C_{f,m})=C_{m,f}$, where we identify $M\to M^{**}$ as v.spaces, i.e. $S^*(\varphi)=\varphi\circ S$ $\forall \varphi\in k_q[G]$.

Let us finally compare this with our definition of $SL_q(2)$, when $\mathfrak{g}=\mathfrak{sl}_2$.

Consider two fin. dim. $U_q(\mathfrak{g})$-modules M, M'. Choose bases $\{e_1, \ldots, e_n\}$ of M, $\{e'_1, \ldots, e'_m\}$ of M', and let $\{f_1, \ldots, f_n\}$, $\{f'_1, \ldots, f'_m\}$ be the dual bases of M^* and $(M')^*$. Set $c_{ij} := c_{f_i, e_j}$, $c'_{ij} := c_{f'_i, e'_j}$. Consider the $U_q(\mathfrak{g})$-isomorphism

$$R = \Theta^f \circ \tau : M' \otimes M \longrightarrow M \otimes M'.$$

Let $\{R_{ij}^{k\ell}\}$ be the matrix coeff-s of R, i.e. $R(e'_i \otimes e_j) = \sum_{k,\ell} R_{ji}^{k\ell}\, e_k \otimes e'_\ell$

<u>Lemma 7</u>: For any i, j, r, s: $\sum_{k,\ell} R_{k\ell}^{rs} c'_{\ell i} c_{kj} = \sum_{k,\ell} R_{ji}^{k\ell} c_{rk} c'_{s\ell}$

▶ $u(e'_i \otimes e_j) = \sum_{k,\ell} c_{f'_\ell \otimes f_k,\, e'_i \otimes e_j}(u) \cdot e'_\ell \otimes e_k \overset{\text{Lemma 6}}{=\!=} \sum_{k,\ell} (c'_{\ell i} c_{kj})(u) \cdot e'_\ell \otimes e_k$

$\Rightarrow R(u(e'_i \otimes e_j)) = \sum_{k,\ell,r,s} R_{k\ell}^{rs} (c'_{\ell i} c_{kj})(u) \cdot e_r \otimes e'_s$

Likewise:

$u(R(e'_i \otimes e_j)) = \sum_{k,\ell} R_{ji}^{k\ell} \cdot u(e_k \otimes e'_\ell) = \sum_{k,\ell,r,s} R_{ji}^{k\ell} \cdot (c_{rk} c'_{s\ell})(u) \cdot e_r \otimes e'_s$ ■

<u>Exercise 7</u>: (a) Apply Lemma 7 in the particular case $\mathfrak{g}=\mathfrak{sl}_2$, $M=M'=L(1,+)$, $R=\Theta^f \circ \tau$ as in Exercise 4 to obtain the following rel-s on $\{c_{ij}\}_{i,j=1}^2$:

$c_{11}c_{12} = q c_{12} c_{11}$, $c_{11}c_{21} = q c_{21}c_{11}$, $c_{12}c_{21} = c_{21}c_{12}$, $c_{12}c_{22} = q c_{22}c_{12}$, $c_{21}c_{22} = q c_{22}c_{21}$,
$c_{11}c_{22} - c_{22}c_{11} = (q-q^{-1}) c_{12}c_{21}$.

(b) Show that $k_q[SL_2]$ is generated by $\{c_{ij}\}_{i,j=1}^2$

(c) Verify $c_{11}c_{22} - q c_{12}c_{21} = 1$.

In particular, we get $SL_{q^{-1}}(2) \twoheadrightarrow k_q[SL_2]$.

(d)* Prove this is an isomorphism $SL_{q^{-1}}(2) \xrightarrow{\sim} k_q[SL_2]$.