<u>Goal</u> (for ~2 classes): Construct a braid group action on $U_q(\mathfrak{g})$ (due to Lusztig), which will ultimately lead to the PBW basis.

<u>Motivation</u>: In the classical case, given a f.d. $\mathfrak{g}$-module V, its weights are W-invariant. Moreover, an explicit operator $V_\mu \to V_{s_\alpha(\mu)}$ representing the simple reflection is given by $\tilde{s}_\alpha = \exp(e_\alpha)\exp(-f_\alpha)\exp(e_\alpha)$, where e_α, f_α denote the endomorphisms of V by which gen-s e_α, f_α act.

In the basic example of the adjoint repr-n $V=\mathfrak{g}$, we get
$$\exp(\mathrm{ad}(e_\alpha))\exp(-\mathrm{ad}(f_\alpha))\exp(\mathrm{ad}(e_\alpha)) = \mathrm{Ad}(\exp(e_\alpha)\exp(-f_\alpha)\exp(e_\alpha)).$$
In the simplest case $\mathfrak{g}=\mathfrak{sl}_2$, this is just $\mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

<u>Exercise 1</u>: Verify that $\tilde{s}_\alpha = \sum\limits_{a,b,c\geq 0} \frac{e_\alpha^a}{a!}\cdot\frac{(-f_\alpha)^b}{b!}\cdot\frac{e_\alpha^c}{c!}$ indeed maps V_μ to $V_{s_\alpha(\mu)}$

<u>Hint</u>: It suffices to verify this for $\mathfrak{g}=\mathfrak{sl}_2$, V – irr. repr-n.

<u>Want</u>: Analogous operators on $U_q(\mathfrak{g})$-modules.

As always, we will first propose a construction for $\mathfrak{g}=\mathfrak{sl}_2$, which will be immediately generalized for an arbitrary $\mathfrak{g}$.

<u>$\mathfrak{g}=\mathfrak{sl}_2$</u>

We shall restrict our attention only to type 1 f.d. $U_q(\mathfrak{sl}_2)$-modules, i.e. $V=\bigoplus_{m\in\mathbb{Z}} V_m$, $V_m = \{v\in V \mid K(v)=q^m\cdot v\}$. For $v\in V_m$, we define its images under the 4 operators $T, T', {}^\omega T, {}^\omega T'$ as follows:

$$\boxed{\begin{aligned} T(v) &:= \sum_{\substack{a,b,c\geq 0\\ -a+b-c=m}} (-1)^b q^{b-ac} E^{(a)}F^{(b)}E^{(c)}v, \quad & T'(v) &:= \sum_{\substack{a,b,c\geq 0\\ -a+b-c=m}} (-1)^b q^{-b+ac} E^{(a)}F^{(b)}E^{(c)}v \\ {}^\omega T(v) &:= \sum_{\substack{a,b,c\geq 0\\ a-b+c=m}} (-1)^b q^{b-ac} F^{(a)}E^{(b)}F^{(c)}v, \quad & {}^\omega T'(v) &:= \sum_{\substack{a,b,c\geq 0\\ a-b+c=m}} (-1)^b q^{-b+ac} F^{(a)}E^{(b)}F^{(c)}v \end{aligned}}$$

<u>Rmks</u>: (1) As E,F act loc. nilpotently on V, each of these 4 sums is finite.

(2) Each of the 4 operators maps V_m to V_{-m}

(3) If we use ${}^\omega V$ to denote a twist of V by the Cartan involution, while ${}^\omega v$ denotes $v\in V$ when viewed as an el-t of ${}^\omega V$, then:
$${}^\omega T({}^\omega v) = {}^\omega(T(v)),\quad {}^\omega T'({}^\omega v) = {}^\omega(T'(v)) \quad \forall v\in V$$

Assuming $q \neq \sqrt{1}$, any fin. dim. $U_q(\mathfrak{sl}_2)$-module is a direct sum of $L(n,+)$. For that reason it suffices to understand how $T, T', {}^{\omega}T, {}^{\omega}T'$ act on $L(n,+)$.
For the simplification of formulas, we will slightly rescale the basis of $L(n,+)$ which was chosen in [Lecture 7, Prop. 1], so that now the basis $\{v_i\}_{i=0}^{n}$ is still a weight basis with $K(v_i) = q^{n-2i} v_i$ and

$$(1)\quad \boxed{F(v_i) = \begin{cases} [i+1]\cdot v_{i+1}, & \text{if } i<n \\ 0, & \text{if } i=n \end{cases} \quad \& \quad E(v_i) = \begin{cases} [n+1-i]\, v_{i-1}, & \text{if } i>0 \\ 0, & \text{if } i=0 \end{cases}}$$

These formulas immediately imply how the divided powers act:

$$(2)\quad \boxed{F^{(r)}(v_i) = \begin{bmatrix} r+i \\ r \end{bmatrix} \cdot v_{i+r} \quad \& \quad E^{(r)}(v_i) = \begin{bmatrix} n+r-i \\ r \end{bmatrix} v_{i-r}, \text{ where } v_{<0} = 0 = v_{>n}}$$

Lemma 1: The following formulas hold:

(a) $T(v_i) = (-1)^{n-i} q^{(n-i)(i+1)} v_{n-i}$

(b) $T'(v_i) = (-1)^{n-i} q^{-(n-i)(i+1)} v_{n-i}$

(c) ${}^{\omega}T(v_i) = (-1)^{i} q^{i(n+1-i)} v_{n-i}$

(d) ${}^{\omega}T'(v_i) = (-1)^{i} q^{-i(n+1-i)} v_{n-i}$

▶ Since $v_i \in L(n,+)_{n-2i}$ and we know the explicit action of divided powers (2), parts (a,b) reduce to the following equality:

$$\sum_{\substack{a,b,c\geq 0 \\ -a+b-c=n-2i}} (-1)^b \cdot q^{\pm(b-ac)} \cdot \begin{bmatrix} n+c-i \\ c \end{bmatrix} \begin{bmatrix} i-c+b \\ b \end{bmatrix} \begin{bmatrix} n+a-i-b+c \\ a \end{bmatrix} = (-1)^{n-i} \cdot q^{\pm(n-i)(i+1)}$$

Note that $\begin{bmatrix} n+a-i-b+c \\ a \end{bmatrix} = \begin{bmatrix} i \\ a \end{bmatrix}$ above as $a-b+c = 2i-n$. We also note that the nontrivial summands may appear only for $c \leq i$, $b \leq n+c-i \overset{a-b+c=2i-n}{\Longrightarrow} a \leq i$.
Hence, if we use j to denote $j = n-i$, the above equality can be rewritten as:

$$(3)\quad \boxed{\sum_{\substack{0\leq a\leq i \\ 0\leq c\leq i}} (-1)^{b+j} \cdot q^{\pm(b-ac-j(i+1))} \cdot \begin{bmatrix} i \\ a \end{bmatrix} \begin{bmatrix} j+c \\ c \end{bmatrix} \begin{bmatrix} j+a \\ i-c \end{bmatrix} = 1}$$

Exercise 2: Prove the equality (3)

Hint: You may want to use (and prove) $\begin{bmatrix} a+b \\ k \end{bmatrix} = \sum_{i=0}^{k} q^{ai-b(k-i)} \begin{bmatrix} a \\ k-i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix}$ (Provide an interpretation in spirit of [HW3, Prob 1])

Modulo this exercise, the proof of (a,b) is complete!

The proofs of (c,d) can be given along the same lines. However, they can be also immediately deduced from (a,b) by using Rmks (3) from p.1

(Continuation of the proof of Lemma 1).

To do that, we note that $^{\omega}L(n,+) \xrightarrow{\sim} L(n,+)$, so that:

$$^{\omega}v_{n-i} \longmapsto v_i$$

$$^{\omega}T(v_i) = {}^{\omega}T({}^{\omega}v_{n-i}) \overset{\text{Rmk(3)}}{=\!=} {}^{\omega}(T(v_{n-i})) \overset{(a)}{=\!=} (-1)^i \cdot q^{(n-i+1)\cdot i} \cdot {}^{\omega}v_i = (-1)^i q^{i(n-i+1)} v_{n-i}$$

and likewise one computes $^{\omega}T'(v_i)$ $\blacksquare$

Corollary 1: For any fin. dim. $U_q(sl_2)$-module V, we have:

(a) $T^{-1} = {}^{\omega}T'$, $(T')^{-1} = {}^{\omega}T$

(b) for $v \in V_m$: $^{\omega}T(v) = (-q)^{-m}T(v)$, $^{\omega}T'(v) = (-q)^m T'(v)$.

As noted right before Lemma 1, it suffices to treat the case $V = L(n,+)$, for which we have explicit formulas of Lemma 1. In particular, all four equalities follow immediately. $\blacksquare$

As a consequence, we see that $T, T', {}^{\omega}T, {}^{\omega}T'$ are actually bijective endom-s.

Lemma 2: For any fin. dim. $U_q(sl_2)$-module V, we have the following "commutation" relations between the operator T and action of generators:

(a) $T(Kv) = K^{-1}T(v)$, $T(Ev) = (-FK)T(v)$, $T(Fv) = (-K^{-1}E)T(v)$

(b) $KT(v) = T(K^{-1}v)$, $FT(v) = T(-EKv)$, $ET(v) = T(-K^{-1}Fv)$

(a) As before, it suffices to consider the case $V = L(n,+)$, $v = v_i$ $(0 \le i \le n)$.

- $Kv_i = q^{n-2i}v_i$, $K^{-1}v_{n-i} = q^{-(n-2(n-i))}v_{n-i} = q^{n-2i}v_{n-i}$, $T(v_i) = \text{const} \cdot v_{n-i} \Rightarrow 1^{\text{st}}$ equality
- $T(Ev_i) \overset{(1)}{=} T([n+1-i]v_{i-1}) \overset{\text{Lemma 1}}{=\!=} (-1)^{n-i+1} \cdot [n+1-i]\, q^{i(n-i+1)} v_{n-i+1}$

  $(-FK)T(v_i) = -(FK)((-1)^{n-i}q^{(i+1)(n-i)}v_{n-i}) = (-1)^{n-i+1} \cdot [n+1-i] \cdot q^{(i+1)(n-i)} \cdot q^{2i-n} \cdot v_{n-i+1}$
- $T(Fv_i) \overset{(1)}{=\!=} [i+1]T(v_{i+1}) = (-1)^{n-i-1} \cdot [i+1] \cdot q^{(i+2)(n-i-1)} v_{n-i-1}$

  $(-K^{-1}E)T(v_i) = (-1)^{n-i-1} \cdot q^{(i+1)(n-i)} K^{-1}E(v_{n-i}) = (-1)^{n-i-1} \cdot [i+1] \cdot q^{(i+1)(n-i)} q^{n-2i-2} v_{n-i-1}$

(b) Follows immediately from (a) $\blacksquare$

Having established the key properties of these four operators $T, T', {}^{\omega}T, {}^{\omega}T'$ (we will concentrate our main attention on T) for the simplest case of sl_2, we are ready to provide the corresponding construction for a general $\mathfrak{g}$.

General $\mathfrak{g}$

For every $\alpha\in\Pi$, we have an embedding $U_{q_\alpha}(\mathfrak{sl}_2)\hookrightarrow U_q(\mathfrak{g})$; in particular, we can apply our construction from $\mathfrak{g}=\mathfrak{sl}_2$ to the pull-back of V along this embedding. In other words, we have operators $\{T_\alpha, T'_\alpha, {}^{\omega}T_\alpha, {}^{\omega}T'_\alpha\}$ whose action on a fin. dim. $U_q(\mathfrak{g})$-module V is given by:

$$\boxed{\begin{aligned} T_\alpha(v) &= \sum_{\substack{a,b,c\geq 0\\ -a+b-c=m}} (-1)^b q_\alpha^{b-ac} E_\alpha^{(a)} F_\alpha^{(b)} E_\alpha^{(c)} v, \quad & T'_\alpha(v) &= \sum_{\substack{a,b,c\geq 0\\ -a+b-c=m}} (-1)^b q_\alpha^{-b+ac} E_\alpha^{(a)} F_\alpha^{(b)} E_\alpha^{(c)} v \\ {}^{\omega}T_\alpha(v) &= \sum_{\substack{a,b,c\geq 0\\ a-b+c=m}} (-1)^b q_\alpha^{b-ac} F_\alpha^{(a)} E_\alpha^{(b)} F_\alpha^{(c)} v, \quad & {}^{\omega}T'_\alpha(v) &= \sum_{\substack{a,b,c\geq 0\\ a-b+c=m}} (-1)^b q_\alpha^{-b+ac} F_\alpha^{(a)} E_\alpha^{(b)} F_\alpha^{(c)} v \end{aligned}}$$

for $v\in V_\lambda$, $m:=\frac{(\lambda,\alpha)}{d_\alpha}=\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$, $E_\alpha^{(a)}:=\frac{E_\alpha^a}{[a]_{q_\alpha}!}$, $F_\alpha^{(a)}:=\frac{F_\alpha^a}{[a]_{q_\alpha}!}$

As an immediate consequence of our discussion for $\mathfrak{g}=\mathfrak{sl}_2$, we obtain:

Corollary 2: (a) $T_\alpha, T'_\alpha, {}^{\omega}T_\alpha, {}^{\omega}T'_\alpha$ are bijective and map V_λ isomorphically onto $V_{s_\alpha(\lambda)}$.

(b) $T_\alpha^{-1}={}^{\omega}T'_\alpha$, $(T'_\alpha)^{-1}={}^{\omega}T_\alpha$

(c) ${}^{\omega}T_\alpha(v)=(-q_\alpha)^{-m}T_\alpha(v)$, ${}^{\omega}T'_\alpha(v)=(-q_\alpha)^{m}T'_\alpha(v)$, where m is as above, $v\in V_\lambda$.

(d) $T_\alpha(E_\alpha v)=(-F_\alpha K_\alpha)T_\alpha(v)$, $T_\alpha(F_\alpha v)=(-K_\alpha^{-1}E_\alpha)T_\alpha(v)$

$E_\alpha T_\alpha(v)=T_\alpha((-K_\alpha^{-1}F_\alpha)v)$, $F_\alpha T_\alpha(v)=T_\alpha((-E_\alpha K_\alpha)v)$

(e) As $(\lambda,\mu)=(s_\alpha\lambda, s_\alpha\mu)$, we immediately see that $T_\alpha(K_\mu v)=K_{s_\alpha\mu}T_\alpha(v)$ $\forall \mu\in Q$.

Q: What about the "commutation" relation b/w T_α and $\{E_\beta, F_\beta\}_{\beta\neq\alpha}$?

If $(\alpha,\beta)=0$, then $[E_\beta,F_\alpha]=0=[E_\beta,E_\alpha]$ & $[F_\beta,F_\alpha]=0=[F_\beta,E_\alpha]$

$\Rightarrow$ $\boxed{T_\alpha(E_\beta v)=E_\beta T_\alpha(v) \ \& \ T_\alpha(F_\beta v)=F_\beta T_\alpha(v) \ \text{ if } (\alpha,\beta)=0}$

Prop 1: For any $\alpha\neq\beta\in\Pi$ and a fin. dim. $U_q(\mathfrak{g})$-module V, $v\in V$, we have:

$$\boxed{T_\alpha(E_\beta v)=\left(\mathrm{ad}(E_\alpha^{(r)})E_\beta\right)T_\alpha(v) \quad \text{with } r:=-\frac{2(\beta,\alpha)}{(\alpha,\alpha)}}$$

Before we sketch the proof, let us discuss some properties of

$\boxed{a_r(m):=\mathrm{ad}(E_\alpha^{(m)})E_\beta,\ m\geq 0}$

• First of all, we recall that according to [Lecture 11, Lemma 3], the Serre rel-n for E_α's is equivalent to $a(r+1)=0$. Hence, $\boxed{a(m)=0 \quad \forall m>r.}$

• Second, due to [Lecture 11, Lemma 2], we have:

$$\boxed{a(m)=\sum_{i=0}^{m} (-1)^i q_\alpha^{i(m-1-r)} E_\alpha^{(m-i)} E_\beta E_\alpha^{(i)}}$$

Lemma 3: $ad(F_\alpha^{(s)})(a(m)) = \begin{cases} \begin{bmatrix} r+s-m \\ s \end{bmatrix}_\alpha \cdot a(m-s), & s\le m \\ 0, & s>m. \end{cases}$

▶ As $ad(F_\alpha^s)=(ad(F_\alpha))^s$ and $ad(F_\alpha)E_\beta=0$, it suffices to prove the above for $s=1$, $m\ge 1$.

According to [Lecture 11, Lemma 1]: $ad(F_\alpha)u=[F_\alpha,u]\cdot K_\alpha$.

$$[F_\alpha, a(m)] = \sum_{i=0}^{m} (-1)^i q_\alpha^{i(m-1-r)}\left(-E_\alpha^{(m-i-1)}[K_\alpha; m-i-1]E_\beta E_\alpha^{(i)} - E_\alpha^{(m-i)} E_\beta E_\alpha^{(i-1)}[K_\alpha; i-1]\right)=$$

$$= \sum_{i=0}^{m-1} (-1)^{i-1} q_\alpha^{i(m-r-1)} E_\alpha^{(m-i-1)} \cdot \frac{K_\alpha q_\alpha^{m-i-1} - K_\alpha^{-1} q_\alpha^{-m+i+1}}{q_\alpha - q_\alpha^{-1}} E_\beta E_\alpha^{(i)} +$$

$$\sum_{i=0}^{m-1} (-1)^i q_\alpha^{(i+1)(m-r-1)} E_\alpha^{(m-i-1)} E_\beta E_\alpha^{(i)} \cdot \frac{K_\alpha q_\alpha^{i} - K_\alpha^{-1} q_\alpha^{-i}}{q_\alpha - q_\alpha^{-1}} =$$

$$= \sum_{i=0}^{m-1} (-1)^i E_\alpha^{(m-1-i)} E_\beta E_\alpha^{(i)} \left(q_\alpha^{(i+1)(m-r-1)} \cdot \frac{K_\alpha q_\alpha^{i} - K_\alpha^{-1} q_\alpha^{-i}}{q_\alpha - q_\alpha^{-1}} - q_\alpha^{i(m-1-r)} \cdot \frac{K_\alpha q_\alpha^{m+i-r-1} - K_\alpha^{-1} q_\alpha^{-m-i+r+1}}{q_\alpha - q_\alpha^{-1}} \right)$$

$$= [r+1-m]_\alpha \cdot a(m-1) K_\alpha^{-1}$$

$\Rightarrow ad(F_\alpha)(a(m)) = [F_\alpha, a(m)]\cdot K_\alpha = [r+1-m]_\alpha \cdot a(m-1)$ ∎

Lemma 4: For any $m, i \ge 0$, we have:

(a) $a(m)E_\alpha^{(i)} = \sum_{j=0}^{i} (-1)^j \begin{bmatrix} m+j \\ j \end{bmatrix}_\alpha q_\alpha^{i(r-2m)-j(i-1)} E_\alpha^{(i-j)} a(m+j)$

(b) $a(m)F_\alpha^{(i)} = \sum_{j=0}^{i} (-1)^j \begin{bmatrix} r-m+j \\ j \end{bmatrix}_\alpha q_\alpha^{j(i-1)} F_\alpha^{(i-j)} a(m-j) K_\alpha^{-j}$

▶ Let us first verify both claims for $i=1$ (as $i=0$ is trivial).

$$\left.\begin{aligned} & ad(E_\alpha)a(m) = [m+1]_\alpha \cdot a(m+1) \\ & \quad \| \text{ [Lecture 11, Lemma 1]} \\ & E_\alpha \cdot a(m) - K_\alpha a(m) K_\alpha^{-1} E_\alpha = E_\alpha \cdot a(m) - q_\alpha^{2m-r} \cdot a(m) E_\alpha \end{aligned}\right\} \Rightarrow \begin{aligned} a(m)E_\alpha = {} & q_\alpha^{r-2m} E_\alpha \cdot a(m) - \\ & q_\alpha^{r-2m} \cdot [m+1]_\alpha \cdot a(m+1) \end{aligned}$$

which is part (a) for $i=1$.

$$\text{By Lemma 3:} \quad \left.\begin{aligned} & ad(F_\alpha)a(m) = [r+1-m]_\alpha \cdot a(m-1) \\ & \quad \| \text{ -//-} \\ & (F_\alpha \cdot a(m) - a(m)F_\alpha)K_\alpha \end{aligned}\right\} \Rightarrow a(m)F_\alpha = F_\alpha a(m) - [r+1-m]_\alpha a(m-1) K_\alpha^{-1}$$

which is part (b) for $i=1$

The case of general $i>1$ is left as an exercise ∎

Exercise 3: Use induction to prove Lemma 4 for $i>1$.

Corollary 3: As $a(>r)=0$, part (a) of Lemma 4 implies

$$\boxed{a(r)E_\alpha^{(i)} = q_\alpha^{-ir} E_\alpha^{(i)} a(r)}$$

Exercise 4: Prove Proposition 1.

Hint: Use Lemma 4 to reduce the equality to a quite simple identity involving q-binomial coefficients.

Corollary 4: For $\alpha \neq \beta \in \Pi$, we have the following equality $\forall v \in V$ – f.d. $U_q(\mathfrak{g})$-mod:

$$\boxed{T_\alpha(F_\beta v) = \left(\sum_{i=0}^{r} (-1)^i q_\alpha^i F_\alpha^{(i)} F_\beta F_\alpha^{(r-i)}\right) T_\alpha(v)}$$

► First of all, using Corollary 2, it is easy to see that if $T_\alpha(uv) = u' T_\alpha(v)$ $\forall v \in V$, then $T_\alpha(\omega(u)v) = (-q_\alpha)^{-\frac{2(\mu,\alpha)}{(\alpha,\alpha)}} \omega(u') T_\alpha(v)$ for $u \in U_q(\mathfrak{g})_\mu$. Hence:

$$T_\alpha(F_\beta v) = T_\alpha(\omega(E_\beta)v) = (-q_\alpha)^r \cdot \omega\left(\sum_{i=0}^{r} (-1)^i q_\alpha^{-i} E_\alpha^{(r-i)} E_\beta E_\alpha^{(i)}\right) T_\alpha(v) =$$

$$= \omega\left(\sum_{i=0}^{r} (-1)^{r-i} \cdot q_\alpha^{r-i} E_\alpha^{(r-i)} E_\beta E_\alpha^{(i)}\right) T_\alpha(v) = \left(\sum_{i=0}^{r} (-1)^i q_\alpha^i E_\alpha^{(i)} E_\beta E_\alpha^{(r-i)}\right) T_\alpha(v)$$ ■

Prop 2: (a) For any $\alpha \in \Pi$ and any $u \in U_q(\mathfrak{g})$, $\exists!\, u' \in U_q(\mathfrak{g})$ such that

$$\boxed{T_\alpha(uv) = u' T_\alpha(v)}$$

for any fin. dim. $U_q(\mathfrak{g})$-module V and $v \in V$.

(b) The assignment $u \mapsto u'$ is an automorphism of $U_q(\mathfrak{g})$.

► (a) According to Corollary 2(d,e), Prop 1, Corollary 4, such u' exist for u chosen to be any of the generators $\{E_\beta, F_\beta, K_\beta^{\pm 1}\}_{\beta \in \Pi}$. Hence, such u' exists for any u.

The uniqueness of u' follows from the fact that T_α is bijective and the fact that any el-t of $U_q(\mathfrak{g})$ acting trivially on all fin. dim. $U_q(\mathfrak{g})$-modules must be ZERO, see [Lecture 12, Theorem 2].

(b) As $(u_1+u_2)' = u_1'+u_2'$, $(u_1u_2)' = u_1'u_2'$, the assignment $u \mapsto u'$ is an algebra endomorphism of $U_q(\mathfrak{g})$. It is injective for the same reason as above.

The surjectivity of this assignment follows from our next exercise ■

Exercise 5: Consider an algebra homomorphism ${}^{\sigma}\mathrm{ad}: U_q(\mathfrak{g}) \to \mathrm{End}_k(U_q(\mathfrak{g}))$
$$x \longmapsto \sigma \circ \mathrm{ad}(x) \circ \sigma : U_q(\mathfrak{g}) \circlearrowleft$$
(a) Prove $({}^{\sigma}\mathrm{ad}(E_\alpha)u)' = \mathrm{ad}(F_\alpha)u'$

(b) Prove $({}^{\sigma}\mathrm{ad}(F_\alpha)u)' = \mathrm{ad}(E_\alpha)u'$

(c) Combine part (a) with Lemma 3 to deduce $\exists u \in U_q(\mathfrak{g})$ s.t. $u' = E_\beta$.

(d) Establish the surjectivity of the assignment $u \mapsto u'$.

Abusing notations, we shall denote the automorphism $u \mapsto u'$ of $U_q(\mathfrak{g})$ from Prop 2 also by T_α, i.e. $u' = T_\alpha(u)$. Then:
$$\boxed{T_\alpha(uv) = T_\alpha(u)T_\alpha(v) \quad \forall u \in U_q(\mathfrak{g}),\ \forall v \in V}$$

Due to Corollary 2(d,e), Prop 1, Corollary 4:
$$T_\alpha: \quad E_\alpha \mapsto -F_\alpha K_\alpha, \quad E_\beta \mapsto \sum_{i=0}^{r} (-1)^i q_\alpha^{-i} E_\alpha^{(r-i)} E_\beta E_\alpha^{(i)} \text{ for } \beta \neq \alpha,\ r = -\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$$
$$F_\alpha \mapsto -K_\alpha^{-1} E_\alpha, \quad F_\beta \mapsto \sum_{i=0}^{r} (-1)^i q_\alpha^{i} F_\alpha^{(i)} F_\beta F_\alpha^{(r-i)} \qquad (*_1)$$
$$K_\mu \mapsto K_{s_\alpha \mu}$$

Due to Exercise 5, we also get explicit formulas for T_α^{-1}:
$$T_\alpha^{-1}: \quad E_\alpha \mapsto -K_\alpha^{-1} F_\alpha, \quad E_\beta \mapsto \sum_{i=0}^{r} (-1)^i q_\alpha^{-i} E_\alpha^{(i)} E_\beta E_\alpha^{(r-i)}$$
$$F_\alpha \mapsto -E_\alpha K_\alpha, \quad F_\beta \mapsto \sum_{i=0}^{r} (-1)^i q_\alpha^{i} F_\alpha^{(r-i)} F_\beta F_\alpha^{(i)} \qquad (*_2)$$
$$K_\mu \mapsto K_{s_\alpha \mu}$$

Comparing formulas $(*_1, *_2)$, we immediately get the following result:

Corollary 5: $\boxed{T_\alpha^{-1} = \sigma \circ T_\alpha \circ \sigma}$

Let us conclude today's lecture by noticing that if $(\alpha,\beta)=0$, then $\{E_\alpha, F_\alpha\}$ commute with $\{E_\beta, F_\beta\}$, and so $\boxed{T_\alpha T_\beta = T_\beta T_\alpha \text{ if } (\alpha,\beta)=0}$

Indeed, this equality viewed inside $\mathrm{End}(V)$ (V-f.d.) is obvious. On the other hand, knowing this equalities for all f.d. $U_q(\mathfrak{g})$-modules. But then this equality also holds in $\mathrm{Aut}(U_q(\mathfrak{g}))$, since $\forall u \in U_q(\mathfrak{g})$ $T_\alpha T_\beta(u) - T_\beta T_\alpha(u)$ acts trivially on all f.d. reps.