—LECTURE 19— (04/16/2018)

Last time: Introduced operators $T_\alpha, T'_\alpha, {}^{\omega}T_\alpha, {}^{\omega}T'_\alpha$ ($\alpha\in\Pi$) acting on any fin.dim. $U_q(\mathfrak{g})$-module, which ultimately give rise to the homonymous automorphisms of $U_q(\mathfrak{g})$. Since they are easily related to each other, we will restrict our attention only to the case of T_α.

The key property of these T_α is due to Lusztig:

Thm: For $\alpha\neq\beta\in\Pi$, let N be the order of $s_\alpha s_\beta\in W$.

(a) The following equality of automorphisms of $U_q(\mathfrak{g})$ holds: $\underbrace{T_\alpha T_\beta T_\alpha\ldots}_{N\text{ times}} = \underbrace{T_\beta T_\alpha T_\beta\ldots}_{N\text{ times}}$

(b) For any fin.dim. $U_q(\mathfrak{g})$-module V, the same equality holds in $\mathrm{End}(V)$.

- Last time we ended the lecture by proving this result in the simplest case $(\alpha,\beta)=0$.
- The original proof of Lusztig establishes the aforementioned relation for modules first, from which the case of $\mathrm{Aut}(U_q(\mathfrak{g}))$ follows easily.

However, in our class we shall only prove (a) directly, but before we do that, let us introduce the general definition.

Def: Given a simple Lie algebra $\mathfrak{g}$ with associated data $(\Pi, (\,,\,)-\text{bilinear pairing on }\Pi)$, one defines the braid group (of type $\mathfrak{g}$) as the group generated by $\{\sigma_\alpha\}_{\alpha\in\Pi}$ subject to the defining relations $\underbrace{\sigma_\alpha\sigma_\beta\sigma_\alpha\ldots}_{N} = \underbrace{\sigma_\beta\sigma_\alpha\sigma_\beta\ldots}_{N}$, where $N=$ order of $s_\alpha s_\beta$ in $W=$ Weyl gp of $\mathfrak{g}$.

Thm 1: The assignment $\sigma_\alpha\mapsto T_\alpha$ gives rise to a braid group action on $U_q(\mathfrak{g})$.

The proof is case-by-case.

Case 1: $(\alpha,\beta)=0$, i.e. $a_{\alpha\beta}=0$.

The equality $T_\alpha T_\beta = T_\beta T_\alpha$ in this case was established last time.

Case 2: $a_{\alpha\beta}=a_{\beta\alpha}=-1$, i.e., $(\alpha,\alpha)=(\beta,\beta)=-2(\alpha,\beta)$

Note that $(\alpha,\alpha)=(\beta,\beta)$ implies $q_\alpha=q_\beta$. Recalling explicit formulas $(*_1,*_2)$ from the end of Lecture 18 for the images of the generators under T_α, T_α^{-1}, we get:

$$\boxed{\begin{aligned} &T_\alpha(E_\beta)=E_\alpha E_\beta - q_\alpha^{-1}E_\beta E_\alpha = T_\beta^{-1}(E_\alpha),\quad T_\beta(E_\alpha)=E_\beta E_\alpha - q_\alpha^{-1}E_\alpha E_\beta = T_\alpha^{-1}(E_\beta),\\ &T_\alpha(F_\beta)=F_\beta F_\alpha - q_\alpha F_\alpha F_\beta = T_\beta^{-1}(F_\alpha),\quad T_\beta(F_\alpha)=F_\alpha F_\beta - q_\alpha F_\beta F_\alpha = T_\alpha^{-1}(F_\beta)\end{aligned}} \quad (1)$$

(Continuation of the proof of Thm 1)

In particular, we immediately get:

$$\boxed{T_\alpha T_\beta(E_\alpha)=E_\beta,\ T_\alpha T_\beta(F_\alpha)=F_\beta,\ T_\beta T_\alpha(E_\beta)=E_\alpha,\ T_\beta T_\alpha(F_\beta)=F_\alpha} \quad (2)$$

Note: One can deduce these equalities by using only the formulas for T_α, T_β (and not for $T_\alpha^{-1}, T_\beta^{-1}$), but this takes a few minutes ($\Leftarrow$ we did that way in class).

Now we are ready to prove the equality $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$ in $\mathrm{Aut}(U_q(\mathfrak{g}))$.

- First, as $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$, we immediately get $T_\alpha T_\beta T_\alpha(K_\mu) = T_\beta T_\alpha T_\beta(K_\mu)\ \forall \mu$.
- Second, due to (2), we have:

$$T_\alpha T_\beta T_\alpha(E_\beta) = T_\alpha(E_\alpha) = -F_\alpha K_\alpha$$
$$T_\beta T_\alpha T_\beta(E_\beta) = T_\beta T_\alpha(-F_\beta K_\beta) = -F_\alpha T_\beta T_\alpha(K_\beta) = -F_\alpha K_\alpha \quad \text{as } s_\beta s_\alpha(\beta)=\alpha.$$
$$T_\alpha T_\beta T_\alpha(F_\beta) = T_\alpha(F_\alpha) = -K_\alpha^{-1} E_\alpha$$
$$T_\beta T_\alpha T_\beta(F_\beta) = T_\beta T_\alpha(-K_\beta^{-1} E_\beta) = -T_\beta T_\alpha(K_{-\beta})\cdot E_\alpha = -K_\alpha^{-1} E_\alpha$$

The other two equalities:

$$T_\alpha T_\beta T_\alpha(E_\alpha) = T_\beta T_\alpha T_\beta(E_\alpha),\quad T_\alpha T_\beta T_\alpha(F_\alpha) = T_\beta T_\alpha T_\beta(F_\alpha)$$

follow by symmetry.

- Finally, we need to verify $\begin{cases} T_\alpha T_\beta T_\alpha(E_\gamma) = T_\beta T_\alpha T_\beta(E_\gamma) \\ T_\alpha T_\beta T_\alpha(F_\gamma) = T_\beta T_\alpha T_\beta(F_\gamma) \end{cases}$ for $\gamma \in \Pi\setminus\{\alpha,\beta\}$.

Since $\mathfrak{g}$ - simple f.d., at least one of the $(\gamma,\alpha), (\gamma,\beta)$ is ZERO.

WLOG we can assume $(\beta,\gamma)=0 \Rightarrow [E_\gamma, E_\beta] = 0 = [E_\gamma, F_\beta] \Rightarrow T_\beta(E_\gamma)=E_\gamma$.

Thus, we need to verify $T_\alpha T_\beta T_\alpha(E_\gamma) = T_\beta T_\alpha(E_\gamma)$. However,

$$\left.\begin{array}{l} [E_\gamma, E_\beta]=0 \Rightarrow [T_\beta T_\alpha(E_\gamma), T_\beta T_\alpha(E_\beta)]=0 \overset{(2)}{\Longrightarrow} [T_\beta T_\alpha(E_\gamma), E_\alpha]=0 \\ [E_\gamma, F_\beta]=0 \Rightarrow \ldots \overset{(2)}{\Longrightarrow} [T_\beta T_\alpha(E_\gamma), F_\alpha]=0 \end{array}\right\} \Rightarrow$$

$\Rightarrow T_\alpha T_\beta T_\alpha(E_\gamma)$ and $T_\beta T_\alpha(E_\gamma)$ act by the same operators on every f.d. $U_q(\mathfrak{g})$-module $\Rightarrow T_\alpha T_\beta T_\alpha(E_\gamma) = T_\beta T_\alpha(E_\gamma) = T_\beta T_\alpha T_\beta(E_\gamma)$ as claimed.

The other equality $T_\alpha T_\beta T_\alpha(F_\gamma) = T_\beta T_\alpha T_\beta(F_\gamma)$ is proved along the same lines.

This completes Case 2.

<u>Case 3</u>: $a_{\beta\alpha}=-2,\ a_{\alpha\beta}=-1$

<u>Exercise 1</u>: Prove $T_\alpha T_\beta T_\alpha T_\beta = T_\beta T_\alpha T_\beta T_\alpha$ — the equality in $\mathrm{Aut}(U_q(\mathfrak{g}))$.

<u>Case 4</u>: $a_{\beta\alpha}=-3,\ a_{\alpha\beta}=-1$.

This case is quite technical. You can try to prove it or see Lusztig's proof ∎

Thus, we have sketched a case-by-case proof of Thm 1.

Rmk: In the above proof we used the fact that $(\gamma,\alpha), (\alpha,\beta), (\gamma,\beta)$ can not all be nonzero for $\mathfrak{g}$-finite simple. However, the original proof of Lusztig eliminates this assumption.

Next, we shall recall some basic results on reduced expressions of $w \in W$. Given $w \in W$, we call a decomposition $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_t}$ reduced if t is minimal, and we define the length of w via $l(w) = t$. The reduced decomposition of w is NOT unique. However, the following is well-known:

Fact ([Bourbaki]): Given two reduced expressions of an element $w \in W$, one can get from one to the other by a sequence of elementary moves, where each such move replaces some subsequence $\underbrace{s_\alpha s_\beta s_\alpha \dots}_{N}$ by $\underbrace{s_\beta s_\alpha s_\beta \dots}_{N}$

Combining this with Thm 1, we can immediately define $T_w \in \mathrm{Aut}(U_q(\mathfrak{g}))$ for any $w \in W$. Indeed, pick any reduced expression $w = s_{\alpha_1} \dots s_{\alpha_t}$ and set $T_w := T_{s_{\alpha_1}} T_{s_{\alpha_2}} \dots T_{s_{\alpha_t}}$. Note that $T_1 = 1$, $T_{s_\alpha} = T_\alpha$.

Likewise, if we take granted the braid gp action on any f.d. $U_q(\mathfrak{g})$-module V, then we can also define $\{T_w\}_{w \in W} \subset \mathrm{End}(V)$ in the same way.

Outcome: We obtain operators $T_w \curvearrowright V$ as well as autom-s $T_w : U_q(\mathfrak{g}) \circlearrowleft$ for all $w \in W$.

Rmks: (1) If $l(w_1 w_2) = l(w_1) l(w_2)$, then $T_{w_1 w_2} = T_{w_1} T_{w_2}$

(2) $T_w(K_\mu) = K_{w(\mu)}$ $\forall \mu \in Q$

$\Downarrow$

$T_w(U_q(\mathfrak{g})_\mu) = U_q(\mathfrak{g})_{w(\mu)}$

(3) $T_w^{-1} = \sigma \circ T_{w^{-1}} \circ \sigma$.

Goal: Use these automorphisms T_w to construct the root vector generators $\{E_\gamma, F_\gamma\}_{\gamma \in \Delta_+}$. (This construction is not unique!)

Lemma 1: Let $\alpha \neq \beta \in \Pi$ and $w \in \langle s_\alpha, s_\beta \rangle \subset W$ be such that $w\alpha > 0$.
(subgp gen-d by s_α, s_β.)

(a) $T_w(E_\alpha) \in \langle E_\alpha, E_\beta \rangle \subset U_q^+$ - the subalgebra of U_q^+ gen-d by E_α, E_β.

(b) $T_w(E_\alpha) = E_{w\alpha}$ if $w\alpha \in \Pi$.

▶ Again, this can be proved case-by-case: $N = 2, 3, 4, 6$ (as before N is the order of $s_\alpha s_\beta$)

- $N=2$
  Then $w\alpha > 0 \Rightarrow w \in \{1, s_\beta\} \Rightarrow T_w(E_\alpha) = E_\alpha = E_{w\alpha}$.

- $N=3$
  Then $w\alpha > 0 \Rightarrow w \in \{1, s_\beta, s_\alpha s_\beta\} \Rightarrow T_w(E_\alpha) \in \{E_\alpha, E_\beta E_\alpha - q_\alpha^{-1} E_\alpha E_\beta, E_\beta\} \Rightarrow$ both claims hold.

- $N=4$
  Exercise 2: Prove both (a,b) in this case.

- $N=6$
  One can try to prove this directly or refer to Lusztig ∎

Prop 1: Let $w \in W$, $\alpha \in \Pi$ satisfy $w\alpha > 0$. Then:

(a) $T_w(E_\alpha) \in U_q^+$

(b) $T_w(E_\alpha) = E_{w\alpha}$ if $w\alpha \in \Pi$.

▶ As the case $w=1$ is trivial, assume $w \in W \setminus \{1\} \Rightarrow \exists \beta \neq \alpha \in \Pi$ s.t. $w\beta < 0$.

Exercise 3: There is a decomposition $w = w'w''$ such that:
(1) $\ell(w) = \ell(w') + \ell(w'')$
(2) $w'' \in \langle s_\alpha, s_\beta \rangle \subset W$
(3) $w'\alpha > 0$, $w'\beta > 0$.

Then, as $w\alpha > 0, w\beta < 0$, we get $w''\alpha > 0, w''\beta < 0$ (explain why) $\overset{w''\neq 1}{\Longrightarrow} \ell(w') < \ell(w)$

According to Lemma 1, $T_{w''}(E_\alpha) \in \langle E_\alpha, E_\beta \rangle$. As $w'\alpha > 0, w'\beta > 0$, by induction assumption $T_{w'}(E_\alpha), T_{w'}(E_\beta) \in U_q^+ \Rightarrow T_w(E_\alpha) = T_{w'}(T_{w''}(E_\alpha)) \in U_q^+ \Rightarrow$ (a).

To prove part (b), we note that $w\alpha \in \Pi \Rightarrow w''\alpha \in \Pi$. Indeed, if it is not so then $w''\alpha = a\cdot\alpha + b\cdot\beta$ $(a, b \in \mathbb{Z}_{>0}) \Rightarrow w\alpha = a \cdot \underbrace{w'(\alpha)}_{>0} + b \cdot \underbrace{w'(\beta)}_{>0}$ is not simple $\Rightarrow$ ↯.
Thus, $w''\alpha \in \Pi$ and by Lemma 1(b) we get $T_{w''}(E_\alpha) = E_{w''\alpha}$. As $\ell(w') < \ell(w)$, the induction assumption applies and we get $T_w(E_\alpha) = T_{w'}(E_{w''\alpha}) = E_{w\alpha}$ ∎

Note: Under the same assumptions, $T_w(F_\alpha) \in U_q^-$ and $T_w(F_\alpha) = F_{w\alpha}$ if $w\alpha \in \Pi$, which is proved in the same way.

In what follows, we will need the following well-known fact:

<u>Fact</u>: Given a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_t}$, we have:

$$\boxed{\{\gamma \in \Delta_+ \mid w^{-1}\gamma < 0\} = \{\alpha_1, s_{\alpha_1}(\alpha_2), s_{\alpha_1} s_{\alpha_2}(\alpha_3), \dots, s_{\alpha_1} \dots s_{\alpha_{t-1}}(\alpha_t)\}}$$

Note that $\forall\, 1 \leq a \leq b \leq t$, $s_{\alpha_a} s_{\alpha_{a+1}} \dots s_{\alpha_b}$ is reduced. Hence, $T_{s_{\alpha_a} \dots s_{\alpha_b}} = T_{\alpha_a} \dots T_{\alpha_b}$.

Due to Prop 1, we get

$$\boxed{T_{\alpha_1} T_{\alpha_2} \dots T_{\alpha_i} (E_{\alpha_{i+1}}) \in U_q^+ \qquad \forall\, 1 \leq i < t}$$

Therefore, for any $a_1, a_2, \dots, a_t \in \mathbb{N}$, the following elements belong to U_q^+:

$$\boxed{T_{\alpha_1} T_{\alpha_2} \dots T_{\alpha_{t-1}} (E_{\alpha_t}^{a_t}) \dots T_{\alpha_1} T_{\alpha_2} (E_{\alpha_3}^{a_3})\, T_{\alpha_1} (E_{\alpha_2}^{a_2})\, E_{\alpha_1}^{a_1}} \quad (\dagger)$$

<u>Prop 2</u>: (a) The products of $(\dagger)$ are linearly independent (for any reduced expression)

(b) The subspace spanned by $(\dagger)$ depends only on w, i.e. it does <u>not</u> depend on the choice of a reduced expression of w.

We will prove this result next time.