# — LECTURE 20 — (04/18/2018)

Last time: $\{T_w\}_{w\in W}\subset \mathrm{Aut}(U_q(\mathfrak{g}))$

Pick $w\in W$ and consider a reduced expression $w=s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_t}$. As recalled last time:

$$\boxed{\{\gamma\in\Delta_+ \mid w^{-1}(\gamma)<0\}=\{\alpha_1, s_{\alpha_1}(\alpha_2),\dots, s_{\alpha_1}\dots s_{\alpha_{t-1}}(\alpha_t)\}}$$

In the end of the previous lecture, we introduced certain el-s of $U_q^+$:

$$(*)\quad \boxed{\left\{T_{\alpha_1}T_{\alpha_2}\dots T_{\alpha_{t-1}}(E_{\alpha_t}^{a_t})\cdot T_{\alpha_1}T_{\alpha_2}\dots T_{\alpha_{t-2}}(E_{\alpha_{t-1}}^{a_{t-1}})\cdot\ldots\cdot T_{\alpha_1}(E_{\alpha_2}^{a_2})\cdot E_{\alpha_1}^{a_1} \mid a_1,\dots,a_t\in\mathbb{Z}_{\geq 0}\right\}}$$

and stated the following result:

**Prop 1**: (a) The products in $(*)$ are linearly independent.

(b) The subspace spanned by all products as in $(*)$ depends only on $w$, i.e. it does not depend on a reduced expression.

▶ (a) The proof is by induction on $t$. Base $t=0$ is obvious. Assume now $t>0$. Note that any product of $(*)$ is of the form $T_{\alpha_1}(x_j)\cdot E_{\alpha_1}^i$, where $x_j$ is a similar product constructed for a reduced expression of $w'=s_{\alpha_1}w$: $w'=s_{\alpha_2}\dots s_{\alpha_t}$. Assume the contrary, i.e. $\sum a_{ij}T_{\alpha_1}(x_j)E_{\alpha_1}^i=0$ for some constants $a_{ij}$. If we can prove that this implies $\sum_j a_{ij}x_j=0\ \forall i$, then the latter implies all $a_{ij}=0$ by induction assumption as $l(w')<l(w)$.

To prove the aforementioned claim, apply $T_{\alpha_1}^{-1}$ to $\sum_i T_{\alpha_1}(\sum_j a_{ij}x_j)E_{\alpha_1}^i=0$ to get $0=\sum_i(\sum_j a_{ij}x_j)T_{\alpha_1}^{-1}(E_{\alpha_1})^i=\sum_i(\sum_j a_{ij}x_j)\cdot(-K_{\alpha_1}^{-1}F_{\alpha_1})^i$. However, as proved in [Lecture 11, Thm 1 and Corollary afterwards] the multiplication map $U_q^+\otimes U_q^0\otimes U_q^-\xrightarrow{\sim}U_q(\mathfrak{g})$ is isom. and $\{F_{\alpha_1}^i\}_{i\geq 0}$ - lin. ind. Hence $\sum_j a_{ij}x_j=0\ \forall i$.

This completes our proof of part (a).

(b) Since the claim is obvious for $w$: $l(w)=0,1$, we can assume $l(w)>1$. As recalled last time, in any two reduced expressions of $w$, the one can be obtained from the other by a sequence of simple moves, which replace a segment $\underbrace{s_\alpha s_\beta\dots}_{N}$ by $\underbrace{s_\beta s_\alpha\dots}_{N}$, $N=$ order of $s_\alpha s_\beta\in W$.

Hence, we can assume that two reduced expressions $w=s_{\alpha_1}\dots s_{\alpha_t}=s_{\beta_1}\dots s_{\beta_t}$ differ just by one simple move and prove the claim for them.

(Continuation of the proof of Prop 1)

If $\alpha_1=\beta_1$, then we can apply the induction assumption to two reduced decompositions of $w'=s_{\alpha_1}w$: $w'=s_{\alpha_2}\ldots s_{\alpha_t}=s_{\beta_2}\ldots s_{\beta_t}$ to immediately obtain the claim.

If $\alpha_t=\beta_t$, then we also get the claim by an induction assumption.

Thus, it suffices to treat the case $\alpha_1\neq\beta_1$, $\alpha_t\neq\beta_t \Rightarrow w=\begin{matrix}s_\alpha s_\beta s_\alpha\ldots\\ =s_\beta s_\alpha s_\beta\ldots\end{matrix}$ (N terms in each)

In this case, the result follows from the following lemma below ∎

<u>Lemma 1</u>: If $\alpha\neq\beta\in\Pi$ and $w$ is the longest element in $\langle s_\alpha,s_\beta\rangle\subset W$ (i.e. $w=\underbrace{s_\alpha s_\beta s_\alpha\ldots}_{N}=\underbrace{s_\beta s_\alpha s_\beta\ldots}_{N}$), then the span of the products in $(*)$ coincides with $\langle E_\alpha,E_\beta\rangle$ — the subalgebra of $U_q^+$ generated by $E_\alpha, E_\beta$.

As in the previous lecture, we utilize a case-by-case argument. There are four options to consider: $N=2,3,4$ or $6$.

- <u>Case 1: $N=2$</u>

  $w=s_\alpha s_\beta=s_\beta s_\alpha$. Due to symmetry, pick the 1st $w=s_\alpha s_\beta$, so that the products in $(*)$ are $T_\alpha(E_\beta^{a_2})E_\alpha^{a_1}=E_\beta^{a_2}E_\alpha^{a_1}$. As $(\alpha,\beta)=0\Rightarrow E_\alpha E_\beta=E_\beta E_\alpha$, we immediately get the result.

- <u>Case 2: $N=3$</u>

  $w=s_\alpha s_\beta s_\alpha=s_\beta s_\alpha s_\beta$. Due to symmetry, consider the first one: $w=s_\alpha s_\beta s_\alpha$. Then, the products of $(*)$ take the form:

$$T_\alpha T_\beta(E_\alpha^{a_3})\,T_\alpha(E_\beta^{a_2})\,E_\alpha^{a_1}=E_\beta^{a_3}\cdot(T_\alpha(E_\beta))^{a_2}\cdot E_\alpha^{a_1}=E_\beta^{a_3}\cdot(E_\alpha E_\beta-q_\alpha^{-1}E_\beta E_\alpha)E_\alpha^{a_1}.$$

In particular, $V:=\mathrm{span}_k\langle(*)\rangle\subseteq U_q^+$. To prove the equality, it suffices to check $V$ is stable under the left multiplication of $E_\alpha, E_\beta$ (as $1\in V$). The claim for $E_\beta$ is obvious. Hence, it remains to show that $E_\alpha\cdot(E_\beta^{a_3}\cdot T_\alpha(E_\beta)^{a_2}\cdot E_\alpha^{a_1})$ can be written as a linear combination of the terms $E_\beta^{?}\cdot T_\alpha(E_\beta)^{?}\cdot E_\alpha^{?}$

<u>Want</u>: to take $E_\alpha$ to the right of $E_\beta^{a_3}$ and then to the right of $T_\alpha(E_\beta)^{a_2}$.

(Continuation of the proof of Lemma 1)

To accomplish the first thing, note that $E_\alpha E_\beta = q_\alpha^{-1} E_\beta E_\alpha + T_\alpha(E_\beta)$, hence we are done if $a_3 \leq 1$. If not, we need to multiply by $E_\beta^{a_3-1}$ on the right.

Note that $T_\alpha(E_\beta)E_\beta = E_\alpha E_\beta^2 - q_\alpha^{-1} E_\beta E_\alpha E_\beta \underset{\text{Serre}}{=} q_\alpha E_\beta E_\alpha E_\beta - E_\beta^2 E_\alpha = q_\alpha E_\beta T_\alpha(E_\beta)$.

So: $E_\alpha E_\beta^2 = q_\alpha^{-1} E_\beta E_\alpha E_\beta + q_\alpha E_\beta T_\alpha(E_\beta) = q_\alpha^{-2} E_\beta^2 E_\alpha + (q_\alpha + q_\alpha^{-1}) E_\beta T_\alpha(E_\beta)$

$$\Downarrow$$

$$E_\alpha E_\beta^3 = q_\alpha^{-3} E_\beta^3 E_\alpha + (q_\alpha^2 + 1 + q_\alpha^{-2}) E_\beta^2 T_\alpha(E_\beta)$$

$$\Downarrow$$

$$\vdots$$

$$\boxed{E_\alpha E_\beta^{a_3} = q_\alpha^{-a_3} E_\beta^{a_3} E_\alpha + [a_3]_\alpha \cdot E_\beta^{a_3-1} T_\alpha(E_\beta)}$$

Hence, we accomplished our first step: moving $E_\alpha$ to the right of $T_\alpha(E_\beta)$.

To move further to the right of $T_\alpha(E_\beta)^{a_2}$, we note that as above:

$$E_\alpha T_\alpha(E_\beta) = E_\alpha^2 E_\beta - q_\alpha^{-1} E_\alpha E_\beta E_\alpha = q_\alpha E_\alpha E_\beta E_\alpha - E_\beta E_\alpha^2 = q_\alpha T_\alpha(E_\beta) E_\alpha.$$

$$\Rightarrow \boxed{E_\alpha T_\alpha(E_\beta)^{a_2} = q_\alpha^{a_2} \cdot T_\alpha(E_\beta)^{a_2} E_\alpha}$$

This completes the proof for $N=3$.

- Case 3: $N=4$

Exercise 1: Prove Lemma 1 for $N=4$

- Case 4: $N=6$

Prove yourself or look at Lusztig's computations ∎

According to Prop 1, $\mathrm{span}_k\langle(*)\rangle$ is independent of the choice of a reduced expression and we will denote it by $U^+[w]$. Recalling the Cartan involution $\omega$ of $U_q(\mathfrak{g})$, we set $U^-[w] := \omega(U^+[w]) \subset U_q^-$

Rmk: The subspaces $U^\pm[w] \subseteq U_q^\pm$ are actually subalgebras, but we will not need this.

Lemma 2: Given $w \in W$, $\alpha \in \Pi$ such that $w^{-1}\alpha < 0$, we have $U^+[w] \cdot E_\alpha \subseteq U^+[w]$.

► As $\#\{\gamma \in \Delta_+ \mid (s_\alpha w)^{-1}\gamma < 0\} < \#\{\gamma \in \Delta_+ \mid w^{-1}\gamma < 0\}$, there is a reduced expression of $w$ of the form $w = s_\alpha s_{\alpha_2} s_{\alpha_3} \ldots s_{\alpha_t}$. But then multiplying any factor of $(*)$ by $E_\alpha$ on the right, we get another factor ∎

Thm 1: Consider any reduced expression of the longest element $w_0$: $w_0 = s_{\alpha_1} \dots s_{\alpha_t}$. Then, all products of $(*)$ form a basis of $U_q^+$.

As $w_0^{-1}\alpha < 0$ $\forall \alpha \in \Pi$, we can apply Lemma 2 to deduce that $U^+[w_0]$ is stable under the right multiplication of $E_\alpha$ $(\alpha \in \Pi)$. As $1 \in U^+[w_0]$ and $U^+[w_0] \subseteq U_q^+$, we actually get the equality $U^+[w_0] = U_q^+$. The claim now follows from Prop 1. ∎

Rmks: (1) Completely analogously, one can also show that the products $\boxed{E_{\alpha_1}^{a_1} T_{\alpha_1}(E_{\alpha_2}^{a_2}) \dots T_{\alpha_1} \dots T_{\alpha_{t-1}}(E_{\alpha_t}^{a_t}) \text{ also form a basis of } U_q^+}$

(2) Likewise, one can also check that the products

$$\boxed{T_{\alpha_1} \dots T_{\alpha_{t-1}}(F_{\alpha_t}^{a_t}) \dots T_{\alpha_1}(F_{\alpha_2}^{a_2}) F_{\alpha_1}^{a_1} \text{ also form a basis of } U_q^-}$$

Note that $\Delta_+ = \{\gamma \in \Delta_+ \mid w_0^{-1}\gamma < 0\} = \{\alpha_1, s_{\alpha_1}(\alpha_2), \dots, s_{\alpha_1} \dots s_{\alpha_{t-1}}(\alpha_t)\}$. Hence, Thm 1 can be viewed as a PBW theorem for $U_q^+$. That accomplishes our goal, i.e. any $\gamma \in \Delta_+$ can be uniquely written as $\gamma = s_{\alpha_1} \dots s_{\alpha_{i-1}}(\alpha_i)$ (for a fixed reduced expression!) and $X_\gamma := T_{\alpha_1} \dots T_{\alpha_{i-1}}(E_{\alpha_i}) \in (U_q^+)_\gamma$ - $q$-analogue of $e_\gamma$ in the classical case.

Warning: This construction of $X_\gamma$ does depend on the choice of a reduced expression for $w_0$.

Remaining Goal: (1) Show that the PBW bases of $U_q^\pm$ are almost dual w.r.t. the pairing $(\cdot,\cdot): U_q^- \times U_q^+ \to k$ of Lectures 14-15.

(2) Replace "TQ condition" by "$q \neq \sqrt{1}$" everywhere.

Let us briefly sketch those results, since we will cover a new topic next time

Recall the pairing $(\cdot,\cdot): U_q^{\leq}\times U_q^{\geq}$ of [Lecture 14, Prop 1]. We also recall the linear maps $r_\alpha, r'_\alpha: U_q^{\pm}\to U_q^{\pm}$ of [Lecture 15, (1–4)] which were shown to satisfy the following properties:

(1) $(F_\alpha y, x)=(F_\alpha,E_\alpha)\cdot(y, r'_\alpha(x))$, $(yF_\alpha, x)=(F_\alpha,E_\alpha)\cdot(y,r_\alpha(x))$

(2) $(y, E_\alpha x)=(F_\alpha,E_\alpha)\cdot(r_\alpha(y),x)$, $(y, xE_\alpha)=(F_\alpha,E_\alpha)\cdot(r'_\alpha(y),x)$

} Lemmas 2(b), 3(b) of Lecture 15

(3) $r_\alpha(xx')=xr_\alpha(x')+q^{(\alpha,\mu')}r_\alpha(x)x'$, $r'_\alpha(xx')=q^{(\alpha,\mu)}xr'_\alpha(x')+r'_\alpha(x)x'$, $x\in(U_q^+)_\mu, x'\in(U_q^+)_{\mu'}$

(4) $r_\alpha(yy')=q^{(\alpha,\mu')}yr_\alpha(y')+r_\alpha(y)y'$, $r'_\alpha(yy')=yr'_\alpha(y')+q^{(\alpha,\mu')}r'_\alpha(y)y'$, $y\in(U_q^-)_{-\mu}, y'\in(U_q^-)_{-\mu'}$

} Lemmas 2(a), 3(a)

Note that formulas (3–4) imply by induction the following equalities:

$$\boxed{r_\alpha(E_\alpha^m)=r'_\alpha(E_\alpha^m)=\frac{q_\alpha^{2m}-1}{q_\alpha^2-1}E_\alpha^{m-1} \quad \& \quad r_\alpha(F_\alpha^m)=r'_\alpha(F_\alpha^m)=\frac{q_\alpha^{2m}-1}{q_\alpha^2-1}F_\alpha^{m-1}}$$

**Lemma 3**: Let $\alpha\in\Pi$. Then $E_\alpha$ is not a zero divisor in $U_q^+$. Moreover, we have:

$$U_q^+=\bigoplus_{i=0}^{\infty} T_\alpha(U^+[s_\alpha w_0])E_\alpha^i, \quad T_\alpha(U^+[s_\alpha w_0])=\{u\in U_q^+ \mid T_\alpha^{-1}(u)\in U_q^+\}.$$

► As in the proof of Thm 1, choose a reduced expression $w_0=s_\alpha s_{\alpha_2}\dots s_{\alpha_t}$. Then right multiplication by $E_\alpha$ maps basis el-s (*) to another basis el-s $\Rightarrow$ $E_\alpha$ is not a right zero divisor. Applying the antiautomorphism $\sigma$, this also implies that $E_\alpha$ is not a left zero divisor either.

Moreover, we also see that $U_q^+=U^+[w_0]=\bigoplus_{i=0}^{\infty}T_\alpha(U^+[s_\alpha w_0])E_\alpha^i$ and clearly $T_\alpha(U^+[s_\alpha w_0])\subseteq V:=\{u\in U_q^+ \mid T_\alpha^{-1}(u)\in U_q^+\}$. The equality $T_\alpha(U^+[s_\alpha w_0])=V$ follows via the same argument we used in the proof of Prop 1(a) (i.e. consider $\underset{\substack{\Uparrow\\ V}}{u}=\sum \underset{\substack{\Uparrow\\ T_\alpha(U^+[s_\alpha w_0])}}{x_i} E_\alpha^i$ and apply $T_\alpha^{-1}$

■

**Corollary**: (1) $U_q^+=T_\alpha(U^+[s_\alpha w_0])\oplus U_q^+E_\alpha$

(2) $U^+[s_\alpha w_0]=\{u\in U_q^+ \mid T_\alpha(u)\in U_q^+\}$ – subalgebra of $U_q^+$

(3) Due to $T_\alpha^{-1}=\sigma\circ T_\alpha\circ\sigma$, we also get $T_\alpha(U^+[s_\alpha w_0])=\sigma(U^+[s_\alpha w_0])$

**Lemma 4**: Let $\alpha\in\Pi$. Then, we have:

(a) $T_\alpha U^+[s_\alpha w_0]=\{x\in U_q^+ \mid r_\alpha(x)=0\}$

(b) $U^+[s_\alpha w_0]=\{x\in U_q^+ \mid r'_\alpha(x)=0\}$

(c) $T_\alpha U^-[s_\alpha w_0]=\{y\in U_q^- \mid r'_\alpha(y)=0\}$

(d) $U^-[s_\alpha w_0]=\{y\in U_q^- \mid r_\alpha(y)=0\}$

We shall only sketch the proof of part (a), since part (b) follows by combining Corollary (3) above with Lemma 2(c) of Lecture 15, while (c-d) are analogous.

"$\subseteq$"

Pick $x \in T_\alpha(U^+[s_\alpha w_0])$. By Lemma 3, we can write $\begin{cases} r_\alpha(x) = (q_\alpha - q_\alpha^{-1}) \sum_{i\geq 0} T_\alpha(u_i) E_\alpha^i \\ r'_\alpha(x) = (q_\alpha - q_\alpha^{-1}) \sum_{i\geq 0} T_\alpha(u'_i) E_\alpha^i \end{cases}$ for some $u_i, u'_i \in U^+[s_\alpha w_0]$.

Recall the equality $xF_\alpha - F_\alpha x = \frac{1}{q_\alpha - q_\alpha^{-1}}\left(r_\alpha(x) K_\alpha - K_\alpha^{-1} r'_\alpha(x)\right)$ of [Lecture 15, Lemma 5] and apply $T_\alpha^{-1}$ to it, to get:

$$\boxed{-T_\alpha^{-1}(x)\cdot E_\alpha K_\alpha + E_\alpha K_\alpha \cdot T_\alpha^{-1}(x) = \sum_{i\geq 0} u_i (-K_\alpha^{-1} F_\alpha)^i K_\alpha^{-1} - \sum_{i\geq 0} K_\alpha \cdot u'_i \cdot (-K_\alpha^{-1} F_\alpha)^i.}$$

But the left-hand side belongs to $U_q^+ \cdot K_\alpha$, while the right-hand side involves terms from $U_q^+ \cdot K_\alpha^{-i-1} F_\alpha^i$ & $U_q^+ \cdot K_\alpha^{-i+1} F_\alpha^i$. Thus, we can easily conclude $u_i = 0\ \forall i$, $u'_i = 0$ for $i > 0$. Thus: $r_\alpha(x) = 0$, $r'_\alpha(x) \in T_\alpha(U^+[s_\alpha w_0])$.

"$\supseteq$"

Pick $x \in U_q^+$ s.t. $r_\alpha(x) = 0$. As $r_\alpha : (U_q^+)_\mu \to (U_q^+)_{\mu-\alpha}$, we can assume $x \in (U_q^+)_\mu$ (for some $\mu$).
By Lemma 3: $x = \sum_{i\geq 0} x_i \cdot E_\alpha^i$ with $x_i \in (T_\alpha U^+[s_\alpha w_0])_{\mu - i\alpha}$. By "$\subseteq$", we have $r_\alpha(x_i) = 0\ \forall i \Rightarrow 0 = r_\alpha(x) = \sum_{i>0} \frac{q_\alpha^{2i}-1}{q_\alpha^2 - 1} x_i E_\alpha^{i-1} \Rightarrow x_i = 0\ \forall i > 0$ (again by Lemma 3) $\Rightarrow x \in T_\alpha U^+[s_\alpha w_0]$.

∎

**Corollary**: $(F_\alpha U_q^-, U^+[s_\alpha w_0]) = 0 = (U_q^- F_\alpha, T_\alpha U^+[s_\alpha w_0])$
$(U^-[s_\alpha w_0], E_\alpha U_q^+) = 0 = (T_\alpha U^-[s_\alpha w_0], U_q^+ E_\alpha)$ (← use (1)&(2) from p.5)

**Lemma 5**: Let $\alpha \in \Pi$, $x \in U^+[s_\alpha w_0]$, $y \in U^-[s_\alpha w_0]$. Then:

(a) $(T_\alpha(y) F_\alpha^i, T_\alpha(x) E_\alpha^i) = (T_\alpha(y), T_\alpha(x)) \cdot (F_\alpha^i, E_\alpha^i)$

(b) $(T_\alpha(y) F_\alpha^j, T_\alpha(x) E_\alpha^i) = 0$ if $j \neq i$.

**Exercise 2**: Prove this result.
Hint: Use induction together with the properties (1-4) from p.5.

**Key THEOREM**: If $\alpha \in \Pi$, $x \in U^+[s_\alpha w_0]$, $y \in U^-[s_\alpha w_0]$, then

$$\boxed{(T_\alpha(y), T_\alpha(x)) = (y, x)}$$

We will not prove this result.

**Prop 2**: Let $w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_t}$ be a reduced expression. Then:

$$\boxed{\begin{aligned}&\left(T_{\alpha_1}\ldots T_{\alpha_{t-1}}(F_{\alpha_t}^{b_t})\cdot\ldots\cdot T_{\alpha_1}(F_{\alpha_2}^{b_2})\cdot F_{\alpha_1}^{b_1},\ T_{\alpha_1}\ldots T_{\alpha_{t-1}}(E_{\alpha_t}^{a_t})\ldots T_{\alpha_1}(E_{\alpha_2}^{a_2})E_{\alpha_1}^{a_1}\right)=\\ &=\begin{cases}0, & \text{if } (b_t,\ldots,b_1)\neq(a_t,\ldots,a_1)\\ \prod_{i=1}^t (F_{\alpha_i}^{a_i}, E_{\alpha_i}^{a_i}), & \text{if } b_i=a_i\ \forall i.\end{cases}\end{aligned}}$$

▶ As always, the proof is by induction on $l(w)$. Base $l(w)=1$ is obvious.
Set $x := T_{\alpha_2}\ldots T_{\alpha_{t-1}}(E_{\alpha_t}^{a_t})\ldots T_{\alpha_2}(E_{\alpha_3}^{a_3})E_{\alpha_2}^{a_2}$, $y := T_{\alpha_2}\ldots T_{\alpha_{t-1}}(F_{\alpha_t}^{b_t})\ldots T_{\alpha_2}(F_{\alpha_3}^{b_3})F_{\alpha_2}^{b_2}$.
Clearly $x\in U^+[s_{\alpha_1}w_0] = \{u\in U_q^+ \mid T_{\alpha_1}(u)\in U_q^+\}$, $y\in U^-[s_{\alpha_1}w_0]$. Thus, the left-hand side of the claimed equality is $(T_{\alpha_1}(y)F_{\alpha_1}^{b_1}, T_{\alpha_1}(x)E_{\alpha_1}^{a_1})$. The claimed equality follows now from Lemma 5 and Key Theorem ■

Finally, let us apply the above results to $w=w_0=s_{\alpha_1}\ldots s_{\alpha_t}$ – reduced expr.
Set $\gamma_i := s_{\alpha_1}\ldots s_{\alpha_{i-1}}(\alpha_i)$, $X_{\gamma_i} := T_{\alpha_1}\ldots T_{\alpha_{i-1}}(E_{\alpha_i})$, $Y_{\gamma_i} := T_{\alpha_1}\ldots T_{\alpha_{i-1}}(F_{\alpha_i})$.
By [Lecture 15, Lemma 1], $(F_{\alpha_i}^r, E_{\alpha_i}^r) = (-1)^r\cdot q_{\alpha_i}^{\frac{r(r-1)}{2}}\cdot \dfrac{[r]_{\alpha_i}!}{(q_{\alpha_i}-q_{\alpha_i}^{-1})^r}$.
Combining this observation with Prop 2 & Thm 1, we see that $\{X_{\gamma_t}^{a_t}\cdot\ldots\cdot X_{\gamma_1}^{a_1}\}_{a_1,\ldots,a_t\geq 0}$ is a basis of $U_q^+$, while $\left\{\prod_{i=1}^t (-1)^{a_i} q_{\alpha_i}^{\frac{-a_i(a_i-1)}{2}} \dfrac{(q_{\alpha_i}-q_{\alpha_i}^{-1})^{a_i}}{[a_i]_{\alpha_i}!}\cdot Y_{\gamma_t}^{a_t}\ldots Y_{\gamma_1}^{a_1}\right\}$ – the dual basis of $U_q^-$ w.r.t. $(\cdot,\cdot)$.

**Corollary**: If $q\neq\sqrt{1}$, then $(\ ,\ ): U_q^-\times U_q^+\to k$ is nondegenerate.

Combining this with "Final Remarks" from Lecture 16, we see that the $\boxed{\text{"TQ condition" can be replaced everywhere by "}q\neq\sqrt{1}\text{".}}$

**Bonus**: We get a product formula for $Ⓗ_\mu$ from Lecture 17:

$$\boxed{\begin{aligned}&Ⓗ_\mu \text{ is the } (U_q^-)_{-\mu}\otimes(U_q^+)_\mu\text{-component of the product}\\ &\qquad Ⓗ^{[t]}\cdot Ⓗ^{[t-1]}\cdot\ldots\cdot Ⓗ^{[2]}\cdot Ⓗ^{[1]},\quad Ⓗ_i=\sum_{r\geq 0}(-1)^r q_{\alpha_i}^{\frac{-r(r-1)}{2}}\cdot\frac{(q_{\alpha_i}-q_{\alpha_i}^{-1})^r}{[r]_{\alpha_i}!}Y_{\gamma_i}^r\otimes X_{\gamma_i}^r\end{aligned}}$$