— LECTURE 21 — (04/23/2018)

* Discuss relation of the PBW basis of $U_q^{\pm}$ to the pairing $(\cdot,\cdot): U_q^{\leq} \times U_q^{\geq} \to k$
(this follows pages 5–7 of the Lecture 20 notes, which we didn't have time to cover in the previous class)

<u>Exercise 0</u>: Given a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_t}$, verify that $\alpha_1, s_{\alpha_1}(\alpha_2), s_{\alpha_1}s_{\alpha_2}(\alpha_3), \to s_{\alpha_1} \dots s_{\alpha_{t-1}}(\alpha_t)$ are pairwise distinct positive roots
[actually, as mentioned before, they are exactly those $\gamma \in \Delta_+$ such that $w^{-1}(\gamma) \in \Delta_-$]

<u>Rmk (1)</u>: It is well-known that for any finite dimensional simple Lie algebra $\mathfrak{g}$, the corresponding braid group admits a topological realization as $\pi_1(\mathfrak{h}^{reg}/W)$, where $\mathfrak{h}^{reg} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta_+} \mathrm{Ker}(\alpha)$ and $\mathfrak{h}$ – Cartan subalgebra.

* * *

<u>Today and next times</u>: Crystal (canonical) bases.

<u>Goal</u>: In any finite dimensional $U_q(\mathfrak{g})$-module construct a basis which is "well-behaved" w.r.t. the action of quantum gp.

Let us start from the following obvious result:

<u>Lemma 1</u>: Given a f.d. $U_q(\mathfrak{g})$-module M, $x \in M_\lambda$, $\alpha \in \Pi$, there exists a unique decomposition $x = \sum_{\substack{j \geq 0 \\ j \geq -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}}} F_\alpha^{(j)} x_j$ s.t. $x_j \in M_{\lambda + j\alpha}$ and $E_\alpha(x_j) = 0$.

▸ Reduces to $\mathfrak{sl}_2$-case, where it is clear ▫

Based on this Lemma, we define the linear operators $\tilde{F}_\alpha, \tilde{E}_\alpha \in \mathrm{End}(M)$ via

$$\boxed{\tilde{F}_\alpha(x) := \sum_{\substack{j \geq 0 \\ j \geq -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}}} F_\alpha^{(j+1)} x_j, \quad \tilde{E}_\alpha(x) := \sum_{\substack{j > 0 \\ j \geq -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}}} F_\alpha^{(j-1)} x_j}$$
with x_j defined in Lemma 1.

<u>Rmks</u>: (1) $F_\alpha M_\lambda = \tilde{F}_\alpha M_\lambda \quad \forall \alpha$

(2) Any $U_q(\mathfrak{g})$-morphism $\varphi: M \to N$ commutes with $\{\tilde{F}_\alpha, \tilde{E}_\alpha\}_{\alpha \in \Pi}$.

From now on, we assume $k = \mathbb{Q}(q)$, q – transcendental / $\mathbb{Q}$.

Define $A \subset k$ via $A = \left\{ \frac{f(q)}{g(q)} \,\middle|\, f, g \in \mathbb{Q}[q], g(0) \neq 0 \right\}$, which is a local ring

<u>Key Point</u>: One can specialize at $q = 0$ when working over A rather than k.

Def 1: Let M be a finite-dimensional $U_q(\mathfrak{g})$-module. An *admissible lattice in M* is an A-submodule $\mathcal{M}$ of M such that

(A1) $\mathcal{M}$ is fin. generated $/A$ and $\mathcal{M}$ generates M over k.

(A2) $\mathcal{M}=\bigoplus_{\lambda\in P}\mathcal{M}_\lambda$, where $\mathcal{M}_\lambda=\mathcal{M}\cap M_\lambda$ (i.e. $\mathcal{M}$-graded).

(A3) $\tilde{E}_\alpha,\tilde{F}_\alpha : \mathcal{M}\circlearrowleft$ $\forall\alpha\in\Pi$.

Def 2: A *crystal base of M* is a pair $(\mathcal{M},\mathcal{B})$, where $\mathcal{M}$ is an admissible lattice in M, while $\mathcal{B}$ is a basis of the $\mathbb{Q}$-vector space $\mathcal{M}/q\mathcal{M}$ s.t.

(C1) $\mathcal{B}=\bigcup_{\lambda\in P}\mathcal{B}_\lambda$, where $\mathcal{B}_\lambda=\mathcal{B}\cap(\mathcal{M}_\lambda/q\mathcal{M}_\lambda)$

(C2) $\tilde{E}_\alpha,\tilde{F}_\alpha:\mathcal{B}\to\mathcal{B}\cup\{0\}$ $\forall\alpha\in\Pi$

(C3) $\forall\, b,b'\in\mathcal{B},\ \alpha\in\Pi$: $b=\tilde{E}_\alpha b' \Leftrightarrow b'=\tilde{F}_\alpha b$.

Rmks: (1) If $\mathcal{M}$ satisfies (A1–A3), then it is a free A-module s.t. $\mathcal{M}\otimes_A k\xrightarrow{\sim}M$.

(2) Given two fin. dim. $U_q(\mathfrak{g})$-modules M_1, M_2, a pair of A-submodules $\mathcal{M}_1\subset M_1$, $\mathcal{M}_2\subset M_2$, and subsets $\mathcal{B}_1\subset\mathcal{M}_1/q\mathcal{M}_1$, $\mathcal{B}_2\subset\mathcal{M}_2/q\mathcal{M}_2$, one can set $\mathcal{M}:=\mathcal{M}_1\oplus\mathcal{M}_2\subset M_1\oplus M_2$ and $\mathcal{B}:=\mathcal{B}_1\times\{0\}\cup\{0\}\times\mathcal{B}_2\subset\mathcal{M}/q\mathcal{M}$.

Then: $(\mathcal{M},\mathcal{B})$ – crystal base of M iff $(\mathcal{M}_i,\mathcal{B}_i)$ – crystal base of M_i for $i=1,2$.

Key Question: Does every fin. dim. $U_q(\mathfrak{g})$-module M admit a crystal base? If yes, then how unique is it?

For the existence, it suffices to treat the case $M=L(\lambda)$. First of all, condition (A2) implies that if $\mathcal{M}$ exists, then it contains some $v_\lambda\in L(\lambda)_\lambda\setminus\{0\}$. But then it must also contain all possible elements $\tilde{F}_{\alpha_1}\cdots\tilde{F}_{\alpha_r}v_\lambda$.

Set $\boxed{\mathcal{L}(\lambda):=A\text{-span of }\{\tilde{F}_{\alpha_1}\cdots\tilde{F}_{\alpha_r}v_\lambda\}}$

Key Theorem: (1) $\mathcal{L}(\lambda)$ is an admissible lattice of $L(\lambda)$.

(2) Moreover, if we set $\mathcal{B}(\lambda)\subset\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ equal to the set of all nonzero images of $\tilde{F}_{\alpha_1}\cdots\tilde{F}_{\alpha_r}v_\lambda$, then $(\mathcal{L}(\lambda),\mathcal{B}(\lambda))$ is a crystal base of $L(\lambda)$.

(3) If $(\mathcal{L},\mathcal{B})$ is a crystal base of $L(\lambda)$, then $\exists\, a\in k^*$ such that $\mathcal{L}=a\cdot\mathcal{L}(\lambda)$ and $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)\xrightarrow[\text{mult. by } a]{\sim}\mathcal{L}/q\mathcal{L}$ maps $\mathcal{B}(\lambda)$ to $\mathcal{B}$.

__Example 1__: Theorem holds for $\mathfrak{g}=\mathfrak{sl}_2$.

__Example 2__: Theorem holds for $\mathfrak{g}$–any, λ–minuscule dominant weight (see Hwk 5, Extra Problems).

__Exercise 1__: Prove both results in Ex1, Ex2.

__Hint__: (1) For $\mathfrak{g}=\mathfrak{sl}_2$, just use a basis $\{v_j\}_{j=0}^{n}$ of $L(n)$ defined via $v_j=F^{(j)}v_\lambda$.

(2) For $\mathfrak{g}$–any, $\lambda\in P_+$–minuscule, apply Problem 2 of "Hwk5, Extra Probl."

(3) You may disregard part (3) of this Thm in the above examples, since it is easy to deduce it in general – next time.

Let us now explain the "global counterpart" of this.
For any $\nu\in Q_+$, we have a surjective map $(U_q^-)_{-\nu}\twoheadrightarrow L(\lambda)_{\lambda-\nu}$ $\forall\lambda\in P_+$
$u\longmapsto uv_\lambda$.
Moreover, for λ "__big enough__" (w.r.t. ν), this map is an isomorphism.
(see our proof of Thm 2 in Lecture 12). Thus, we can pull-back $\mathcal{L}(\lambda)_{\lambda-\nu}$ to obtain $\mathcal{L}(\infty)_{-\nu}\subset(U_q^-)_{-\nu}$ (which is __independent of big λ__), and then pull-back the basis $\mathcal{B}(\lambda)_{\lambda-\nu}$ of $\mathcal{L}(\lambda)_{\lambda-\nu}/q\mathcal{L}(\lambda)_{\lambda-\nu}$ to obtain a basis $\mathcal{B}(\infty)_{-\nu}$ of $\mathcal{L}(\infty)_{-\nu}/q\mathcal{L}(\infty)_{-\nu}$ (also __independent of big λ__)
Thus, we end up with a pair $(\mathcal{L}(\infty),\mathcal{B}(\infty))$, where

$$\boxed{\mathcal{L}(\infty):=\bigoplus_{\nu\in Q_+}\mathcal{L}(\infty)_{-\nu}\subset U_q^-\ ,\ \mathcal{B}(\infty):=\bigcup_{\lambda\in P_+}\mathcal{B}(\infty)_{-\nu}\subset\mathcal{L}(\infty)/q\mathcal{L}(\infty).}$$

__Theorem__: For any $\lambda\in P_+$, $\mathcal{L}(\infty)$ maps onto $\mathcal{L}(\lambda)$, while $\mathcal{B}(\infty)_{-\nu}$ maps to $\mathcal{B}(\lambda)_{\lambda-\nu}\cup\{0\}$.

Moreover, these $\mathcal{L}(\infty),\mathcal{B}(\infty)$ can be defined similarly to $\mathcal{L}(\lambda),\mathcal{B}(\lambda)$.
However, to spell it out, we need to introduce the operators $\widetilde{E}_\alpha,\widetilde{F}_\alpha: U_q^-\circlearrowleft$ in a way similar to the module case. The definition is based on the following counterpart of Lemma 1:

__Lemma 2__: Let $\alpha\in\Pi$ and $y\in U_q^-$. There are __uniquely__ determined elements $y_n\in U_q^-$, almost all ZERO, s.t. $\boxed{r'_\alpha(y_n)=0}$ and $\boxed{y=\sum_{n\geq 0}F_\alpha^{(n)}y_n}$
$(q\mp\sqrt{q})$

__Exercise 2__: Prove Lemma 2 (which is analogous to [Lecture 20, Lemma 3])

Based on Lemma 2, we now define linear operators $\tilde{F}_\alpha, \tilde{E}_\alpha : U_q^- \circlearrowleft$ via

$$\boxed{\tilde{F}_\alpha y := \sum_{n\geq 0} F_\alpha^{(n+1)} y_n, \quad \tilde{E}_\alpha y := \sum_{n\geq 0} F_\alpha^{(n-1)} y_n}$$

Rmk: Clearly $\tilde{F}_\alpha$ — injective.

Then:
$$\boxed{\begin{array}{l} \mathcal{L}(\infty) = A\text{-span of } \{\tilde{F}_{\alpha_1} \ldots \tilde{F}_{\alpha_r} 1\}. \\ \mathcal{B}(\infty) = \text{images of } \tilde{F}_{\alpha_1} \ldots \tilde{F}_{\alpha_r} \mathbb{1} \text{ in } \mathcal{L}(\infty)/q\mathcal{L}(\infty). \end{array}}$$

We shall also discuss Lusztig's approach to $\mathcal{B}(\infty)$, closely related to the PBW basis from the last 2 lectures.

More details & Proofs — next time.