— LECTURE 22 — (04/25/2018)

Last time: We discussed the crystal bases and formulated the main result regarding the crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ of $L(\lambda)$. We also mentioned that they give rise to a certain "universal pair" $(\mathcal{L}(\infty), \mathcal{B}(\infty))$ with $\mathcal{L}(\infty) \subset U_q^-$, $\mathcal{B}(\infty) \subset \mathcal{L}(\infty)/q\mathcal{L}(\infty)$. This is Kashiwara's approach.

Today: We shall construct this "universal" pair for U_q first and then verify that it gives rise to crystal bases $\forall \lambda \in P_+$. This approach is due to Lusztig.

For simplicity of exposition, we shall assume

Assume: • $\mathfrak{g}$ is simply-laced (ADE type).
• base field is $\mathbb{Q}(q)$.

Recall the automorphisms $T_i \in \mathrm{Aut}(U_q(\mathfrak{g}))$ of Lectures 18-20, which satisfy braid relations, where we use indices i instead of α, so that

$$\boxed{T_i: E_j \mapsto \begin{cases} -F_j K_j, & i=j \\ E_i E_j - q^{-1} E_j E_i, & i-j \\ E_j, & \text{otherwise} \end{cases}, \quad F_j \mapsto \begin{cases} -K_j^{-1} E_j, & i=j \\ F_j F_i - q F_i F_j, & i-j \\ F_j, & \text{otherwise} \end{cases}, \quad K_j \mapsto \begin{cases} K_j^{-1}, & i=j \\ K_i K_j, & i-j \\ K_j, & \text{otherwise} \end{cases}}$$

Fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_N}$ and set $\mathbf{i} := (i_1, i_2, \dots, i_N)$, $\beta_k := s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$, so that $\{\beta_1, \dots, \beta_N\} = \Delta_+$. Define

$$\boxed{F_{\mathbf{i},\beta_1} := F_{i_1}, \quad F_{\mathbf{i},\beta_2} := T_{i_1}(F_{i_2}), \ \dots, \ F_{\mathbf{i},\beta_N} := T_{i_1} \dots T_{i_{N-1}}(F_{i_N})}$$

For any $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{Z}_{\geq 0}^N$, set $F_{\mathbf{i}}^{\mathbf{a}} := F_{\mathbf{i},\beta_1}^{(a_1)} \dots F_{\mathbf{i},\beta_N}^{(a_N)}$ with $x^{(a)} := \frac{x^a}{[a]_q!}$.

According to [Lecture 20, Thm 1] or rather its counterpart for U_q^-, we have:

PBW THM: For any $\mathbf{i}$, $B_{\mathbf{i}} := \{F_{\mathbf{i}}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}_{\geq 0}^N\}$ is a $\mathbb{Q}(q)$-basis of U_q^-.

We also have to recall some of the properties of these $F_{\mathbf{i},\beta}$.

Lemma 1: Fix $\mathbf{i}$.

(a) If i_k and i_{k+1} are not adjacent, then reversing their order gives another reduced expression $\mathbf{i}'$, and the root vectors do not change, more precisely, $\beta'_k=\beta_{k+1}$, $\beta'_{k+1}=\beta_k$, $\beta'_l=\beta_l$ for $l\neq k,k+1$.

(b) If $\beta_k=\alpha_j$ for some $1\leq k\leq N$, $1\leq j\leq n$, then $F_{\mathbf{i},\beta_k}=F_j$.

(c) If $i_k=i_{k+2}$ is adjacent to i_{k+1}, then $\beta_{k+1}=\beta_k+\beta_{k+2}$, and:

$$\boxed{F_{\beta_{k+1}}=F_{\beta_{k+2}}F_{\beta_k}-qF_{\beta_k}F_{\beta_{k+2}}}$$

Moreover, for the new reduced expression $\mathbf{i}'$ obtained from $\mathbf{i}$ by replacing $i_k i_{k+1} i_k$ with $i_{k+1} i_k i_{k+1}$, we have $F_{\mathbf{i},\beta}=F_{\mathbf{i}',\beta}$ for $\beta\neq\beta_{k+1}$.

► (a) The first claim is clear as $s_{i_k}s_{i_{k+1}}=s_{i_{k+1}}s_{i_k}$ under the assumption. The second claim also follows as $s_{i_k}(\alpha_{i_{k+1}})=\alpha_{i_{k+1}}$, $s_{i_{k+1}}(\alpha_{i_k})=\alpha_{i_k}$.

(b) This is a suitable version of [Lecture 19, Prop 1(b)]

(c) Applying either $s_{i_{k-1}}^{-1}\ldots s_{i_1}^{-1}$ or $T_{i_{k-1}}^{-1}\ldots T_{i_1}^{-1}$ we reduce both claims to the $k=1$ case, hence, to the $U_q(\mathfrak{sl}_3)$-case. The first equality is clear while the second follows as in [Lecture 20, Lemma 1].

Indeed, $\beta_1=\alpha_1$, $\beta_2=s_{\alpha_1}(\alpha_2)=\alpha_1+\alpha_2$, $\beta_3=s_{\alpha_1}s_{\alpha_2}(\alpha_1)=\alpha_2$ and

$$T_{\mathbf{i},\beta_1}=F_{\alpha_1},\quad T_{\mathbf{i},\beta_3}=F_{\alpha_2},\quad T_{\mathbf{i},\beta_2}=F_{\alpha_2}F_{\alpha_1}-qF_{\alpha_1}F_{\alpha_2}$$

Same computation also explains the equality $F_{\mathbf{i},\beta}=F_{\mathbf{i}',\beta}$ for $\beta\neq\beta_{k+1}$. ∎

Lemma 2: (a) If $1\leq j<k\leq N$, then $T_{i_j}^{-1}\cdots T_{i_1}^{-1}(F_{\mathbf{i},\beta_k})\in U_q^-$

(b) If $1\leq k\leq j\leq N$, then $T_{i_j}^{-1}\ldots T_{i_1}^{-1}(F_{\mathbf{i},\beta_k})\in U_q^{\geq}$

► (a) $T_{i_j}^{-1}\cdots T_{i_1}^{-1}(F_{\mathbf{i},\beta_k})=T_{i_{j+1}}\cdots T_{i_{k-1}}(F_{i_k})\in U_q^-$, compare to [Lecture 19, Prop 1]
(as $s_{i_{j+1}}\ldots s_{i_{k-1}}s_{i_k}$ - reduced $\Rightarrow s_{i_{j+1}}\ldots s_{i_{k-1}}(\alpha_{i_k})$ - positive)

(b) $T_{i_j}^{-1}\ldots T_{i_1}^{-1}(F_{\mathbf{i},\beta_k})=T_{i_j}^{-1}\cdots T_{i_{k+1}}^{-1}\underbrace{T_{i_k}^{-1}(F_{i_k})}_{-K_{i_k}^{-1}E_{i_k}}\in U_q^{\geq}$ via similar arguments ∎

Finally, we shall need certain convexity property of $F_{i,\beta}$:

Lemma 3: Fix i, $1\le j<k\le N$.

(a) Write $F_{\beta_k}F_{\beta_j}=\sum_a p_a\cdot F_i^a$. If $p_a\neq 0$, then $a_\ell=0$ unless $j\le \ell\le k$.

(b) If $n\beta_\ell = a_j\beta_j+\ldots+a_k\beta_k$ for some $n, a_j, a_k>0$, $a_{j+1},\ldots,a_{k-1}\ge 0$, then $j<\ell<k$

▸ (a) By Lemma 2:

$$T_{i_{j-1}}^{-1}\ldots T_{i_1}^{-1}(F_{\beta_k}F_{\beta_j})\in U_q^-,\quad T_{i_k}^{-1}\ldots T_{i_1}^{-1}(F_{\beta_k}F_{\beta_j})\in U_q^{\ge}.$$

On the other hand, F_i^a are lin. indep. and

$$T_{i_{j-1}}^{-1}\ldots T_{i_1}^{-1}(F_i^a)\in U_q^-,\ T_{i_k}^{-1}\ldots T_{i_1}^{-1}(F_i^a)\in U_q^{\ge}\overset{\text{Lemma 2}}{\Longleftrightarrow} a_\ell=0 \text{ unless } j\le\ell\le k$$

(b) Similar argument:

$$s_{i_{j-1}}\ldots s_{i_1}(a_j\beta_j+\ldots+a_k\beta_k)\in \underset{\bigoplus_i \mathbb{Z}_{\ge 0}\alpha_i}{\underset{\parallel}{Q_+}},\quad s_{i_k}\ldots s_{i_1}(a_j\beta_j+\ldots+a_k\beta_k)\in Q_-\Rightarrow$$

$\Rightarrow j\le\ell\le k$.

If $\ell=j$, then $n>a_j$ for degree reasons $\Rightarrow$ apply same argument to $(n-a_j)\beta_j=a_{j+1}\beta_{j+1}+\ldots+a_k\beta_k$ to get contradiction.

The case $\ell=k$ is treated in a similar way.

So: $j<\ell<k$. ∎

Now we are ready to start proving main results. Define

$$\boxed{\mathcal{L}:=\operatorname{Span}_{\mathbb{Z}[q]}B_i}$$

for a fixed choice of a reduced expression i.

Thm 1: (a) $\mathcal{L}$ is independent of i

(b) The basis $B_i+q\mathcal{L}$ of $\mathcal{L}/q\mathcal{L}$ is independent of i.

▸ As any reduced expression is obtained from another by a sequence of simple (braid) moves, it suffices to consider i' obtained from i by a single braid move.

If this is a 2-term braid move, then the result is clear as $\{F_{i,\beta}\}_{\beta\in\Delta_+}=\{F_{i',\beta}\}_{\beta\in\Delta_+}$.

It remains to consider a 3-term braid move involving $i_k, i_{k+1}, i_{k+2}=i_k$.

(Continuation of the proof of Thm 1).

Note that $F_{i,\beta} = F_{i',\beta}$ unless $\beta = \beta_{k+1}$ and moreover $\beta_\ell = \beta'_\ell$ for $\ell \neq k, k+1, k+2$. Hence, the claim essentially reduces to the $\mathfrak{g} = \mathfrak{sl}_3$ case. For the latter case, let α_1, α_2 be the two simple roots of $\mathfrak{sl}_3$ and $i = (1,2,1)$, $i' = (2,1,2)$. As we recalled in the proof of Lemma 1: $F_{i,\alpha_2} = F_{\alpha_2}$, $F_{i',\alpha_1} = F_{\alpha_1}$ and thus the claim of thm reduces to:

$(*_1)$ $\mathrm{Span}_{\mathbb{Z}[q]} \{ F_{\alpha_1}^{(a_1)} T_{\alpha_1}(F_{\alpha_2})^{(a_2)} F_{\alpha_2}^{(a_3)} \,|\, a_1, a_2, a_3 \in \mathbb{Z}_{\geq 0} \} = \mathrm{Span}_{\mathbb{Z}[q]} \{ F_{\alpha_2}^{(b_1)} T_{\alpha_2}(F_{\alpha_1})^{(b_2)} F_{\alpha_1}^{(b_3)} \,|\, b_1 \geq 0, b_2 \geq 0, b_3 \geq 0 \}$

$(*_2)$ The sets of $(*_1)$ do coincide modulo q.

Exercise 1: Verify $(*_1, *_2)$.

Hint: Use the formulas from the proof of [Lecture 20, Lemma 1]

Rmk: One may ask how a' is related to a, once we identify F_i^a with $F_{i'}^{a'}$ modulo q, for i' obtained from i by a single braid move.

If the braid move was 2-term (involving i_k, i_{k+1}), then $a'_\ell = a_\ell$ for $\ell \neq k, k+1$, while $a'_k = a_{k+1}$, $a'_{k+1} = a_k$.

If that was a 3-term braid move involving i_k, i_{k+1}, i_{k+2}, then clearly $a'_\ell = a_\ell$ for $\ell \neq k, k+1, k+2$, while the other 3 are related via

$$(*_3) \quad \boxed{\begin{aligned} a'_k &= \max\{a_{k+1}, a_{k+1} + a_{k+2} - a_k\} \\ a'_{k+1} &= \min\{a_k, a_{k+2}\} \\ a'_{k+2} &= \max\{a_{k+1}, a_{k+1} + a_k - a_{k+2}\} \end{aligned}}$$

Exercise 2: Verify the formula $(*_3)$

KEY CONSTRUCTION: Consider the so-called "bar-involution" of the $\mathbb{Q}$-algebra $U_q(\mathfrak{g})$ defined via

$$\boxed{\overline{E_i} = E_i,\ \overline{F_i} = F_i,\ \overline{K_i} = K_i^{-1},\ \overline{q} = q^{-1}}$$

We shall see now how PBW base behaves under this bar-involution.

First, let us introduce the following partial order "$\preccurlyeq$" on the sequences $a=(a_1,\dots,a_N)\in\mathbb{Z}_{\geq 0}^N$ (as before, we fix $\mathbf{i}$ first).

Def: We shall say $a\preccurlyeq a'$ if the following 3 conditions hold:

(1) $wt(a)=wt(a')$, where $wt(a):=\sum_{i=1}^N a_i\beta_i$.

(2) $a'_1>a_1$, or $a'_1=a_1$ & $a'_2>a_2$, or $a'_1=a_1$ & $a'_2=a_2$ & $a'_3\geq a_3$, ---

(3) $a'_N>a_N$, or $a'_N=a_N$ & $a'_{N-1}>a_{N-1}$, or $a'_N=a_N$ & $a'_{N-1}=a_{N-1}$ & $a'_{N-2}>a_{N-2}$, or...

We shall say $a\prec a'$ if $a\preccurlyeq a'$ & $a\neq a'$.

<u>Thm 2</u>: Fix $\mathbf{i}$ and pick any $a\in\mathbb{Z}_{\geq 0}^N$. Then:

$$\boxed{\overline{F_{\mathbf{i}}^a}=F_{\mathbf{i}}^a+\sum_{a'\prec a}p_{a'}^a(q)\cdot F_{\mathbf{i}}^{a'}\ ,\quad p_{a'}^a(q)\text{ -- Laurent polynomial in } q.}$$

- The fact that $p_{a'}^a(q)$ are Laurent pol-s is quite clear from our construction.
- Key observation is that due to Lemma 3(a), it suffices to verify the statement in the simplest case when $a_k=1$ & $a_l=0$ for $l\neq k$, i.e. $F_{\mathbf{i}}^a=F_{\mathbf{i},\beta}$, which shall occupy the last step.
- If β - simple $\Rightarrow$ $\overline{F_{\mathbf{i},\beta}}=F_{\mathbf{i},\beta}$ $\Rightarrow$ statement obviously holds.

  Otherwise, we will prove by an induction on $ht(\beta)$ $\left(\begin{array}{l}\text{If } \beta=\sum_{i=1}^n m_i\alpha_i\\ ht(\beta):=\sum m_i\end{array}\right)$

  First, as $F_{\mathbf{i},\beta}$ is the maximal elt of weight β, we have $\overline{F_{\mathbf{i},\beta}}=\sum_{a'\preccurlyeq a=(0,\dots,1,\dots,0)} p_{a'}^a(q)\cdot F_{\mathbf{i}}^{a'}$.

  <u>Remains</u>: $p_a^a(q)=1$.

Pick a simple root α_j, s.t. $(\alpha_j,\beta)<0$. There exist two reduced expressions $\mathbf{i}',\mathbf{i}''$ of w_0 such that $\beta'_1=\alpha_j$ and $\beta''_N=\alpha_j$. Hence, there is a sequence of braid moves moving $\mathbf{i}$ either to $\mathbf{i}'$ or $\mathbf{i}''$ so that α_j is moved past β. It is straightforward to see that braid moves that do not change β map terms $\prec F_\beta$ to linear combination of terms $\prec F_\beta$, thus, $p_a^a(q)$ - does not change. At the next move which changes finally F_β, we can apply Lemma 1(b) to write (assuming that braid move involves spots $k,k+1,k+2$, so that $\beta=\beta_{k+1}$):

$$F_{\beta_{k+1}}=F_{\beta_{k+2}}F_{\beta_k}-qF_{\beta_k}F_{\beta_{k+2}}\Rightarrow \overline{F_{\beta_{k+1}}}=\overline{F_{\beta_{k+2}}}\,\overline{F_{\beta_k}}-q^{-1}\overline{F_{\beta_k}}\,\overline{F_{\beta_{k+2}}}\Rightarrow$$

$$\Rightarrow\boxed{\overline{F_{\beta_{k+1}}}-F_{\beta_{k+1}}=(\overline{F_{\beta_{k+2}}}-F_{\beta_{k+2}})F_{\beta_k}+\overline{F_{\beta_{k+2}}}(\overline{F_{\beta_k}}-F_{\beta_k})+qF_{\beta_k}F_{\beta_{k+2}}-q^{-1}\overline{F_{\beta_k}}\,\overline{F_{\beta_{k+2}}}}$$

<u>But</u>: $\beta_{k+1}=\beta_k+\beta_{k+2}$ $\Rightarrow$ induction applies to $\overline{F_{\beta_k}}-F_{\beta_k}$ and $\overline{F_{\beta_{k+2}}}-F_{\beta_{k+2}}$.

Combining this with $\beta_{k+1}-\beta_{k+2}\in Q_+$ and Lemma 3, the equality $p_a^a(q)=1$ follows ∎

THM3: There is a unique basis $\mathcal{B}$ of $U_q(\mathfrak{g})$ such that:

(a) $\mathcal{B}\subset\mathcal{L}$, $\mathcal{B}+q\mathcal{L}$ is a basis of $\mathcal{L}/q\mathcal{L}$ that agrees with all $B_i+q\mathcal{L}$.

(b) $\mathcal{B}$ is bar-invariant.

Moreover, the change of basis from any B_i to $\mathcal{B}$ is unit-upper-triangular.

▸ If a is minimal w.r.t. "$<$", then by Lemma 3(b) $a_k\neq 0 \Rightarrow \beta_k$-simple. Hence, in this case $\overline{F^a}=F^a$ as $\overline{F_\alpha}=F_\alpha$ for simple roots α.

Let us now proceed by induction on the partial order. Fix a non-minimal a and assume we already established the result for $a'<a$. Then together with Thm 2, that implies we can write:

$$\boxed{\overline{F^a}=F^a+\sum_{a'<a}\underbrace{f^a_{a'}(q)}_{\text{Laurent pol-l in } q}\cdot \underbrace{b^{a'}}_{\text{inductively determined bar-invariant el-s}}}$$

Then: $F^a=\overline{\overline{F^a}}=F^a+\sum_{a'<a}\left(f^a_{a'}(q)+f^a_{a'}(q^{-1})\right)b^{a'} \Rightarrow f^a_{a'}(q)+f^a_{a'}(q^{-1})=0\ \forall a'<a$

Hence, $\forall a'$ $\exists$ polynomial $g^a_{a'}(q)$ s.t. $\boxed{f^a_{a'}(q)=q\cdot g^a_{a'}(q)-q^{-1}g^a_{a'}(q^{-1})}$

Set: $b^a:=F^a+\sum_{a'<a}q\cdot g^a_{a'}(q)\,b^{a'}$

Then, $\overline{b^a}=F^a+\sum_{a'<a}\left(f^a_{a'}(q)+q^{-1}g^a_{a'}(q^{-1})\right)b^{a'}=F^a+\sum_{a'<a}q\cdot g^a_{a'}(q)\,b^{a'}=b^a$, completing the step of induction. We note that replacing F^a by b^a does not change $\mathcal{L}$ and $b^a\equiv F^a$ modulo $q\mathcal{L}$. ∎

Def: The basis $\mathcal{B}=\{b^a\}_{a\in\mathbb{Z}^N_{\geq 0}}$ is the Lusztig's canonical basis.

Let us finally relate this to the discussion from the previous Lecture. We start from the following simple result:

Prop 1: Fix $\lambda\in P_+$ and let us write the irreducible module $L(\lambda)$ as $L(\lambda)=U_q/I_\lambda$. Then $\mathcal{B}\cap I_\lambda$ spans I_λ. In other words, $\{b+I_\lambda \mid b\in\mathcal{B}, b\notin I_\lambda\}$ is a basis of $L(\lambda)$.

▸ According to [Lecture 13, Thm 1], $L(\lambda)\simeq\bar{L}(\lambda)\Rightarrow I_\lambda=\sum_i U_q\cdot F_i^{m_i+1}$, where $\lambda=\sum_i m_i\omega_i$. Hence, suffices to show $\mathcal{B}\cap U_qF_i^n$ spans $U_qF_i^n$ $\forall n\geq 0$.

This immediately follows from the PBW thm with $\mathbf{i}$ chosen so that $\beta_N=\alpha_i$, combined with the fact that change of basis $B_{\mathbf{i}}\rightsquigarrow\mathcal{B}$ is upper-triangular ∎

Recall [Lecture 15, Lemma 5] which says that $\forall y \in U_q^-$:

$$\boxed{E_i y - y E_i = \frac{1}{q-q^{-1}}\left(K_i r_i(y) - r_i'(y) K_i^{-1}\right)}$$

And as we saw last time, see [Lecture 21, Lemma 2], $y \in U_q^-$ can be uniquely written as $y = \sum_{n\geq 0} F_i^{(n)} y_n$ with $r_i'(y_n)=0$. Last time we defined $\tilde{F}_i(y)$ via

$$\boxed{\tilde{F}_i(y) = \sum_{n\geq 0} F_i^{(n+1)} y_n}$$

We introduce $\mathcal{L}(\infty)$ similarly to $\mathcal{L}(\lambda)$ last time, i.e. we set $\mathcal{L}(\infty)$ to be the A-span $\langle \tilde{F}_{i_1} \dots \tilde{F}_{i_k} 1 \rangle$, where $A = \{\frac{f(q)}{g(q)} \mid f,g \in \mathbb{Q}[q],\ g(0)\neq 0\}$. There is a unique basis $\mathcal{B}(\infty)$ of $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$ s.t. $\tilde{F}_i$ act by permuting el-s of $\mathcal{B}(\infty)$.

Thm 4: $\mathcal{L}(\infty) = \mathrm{span}_A \mathcal{B}$, $\mathcal{B}(\infty) = \mathcal{B} + q\mathcal{L}(\infty)$

▶ Fix $1\leq i \leq n$ and choose $\mathbf{i}$ s.t. $i_1 = i$.

First, similarly to [Lecture 20, Lemma 3], $\mathrm{Ker}(r_i') = \mathrm{span}\{F_{\mathbf{i},\beta_2}^{(a_2)} \dots F_{\mathbf{i},\beta_N}^{(a_N)}\}$. In particular, $\tilde{F}_i$ acts by partial permutation of $\mathcal{B}_{\mathbf{i}}$. Hence, $\mathrm{span}_A \mathcal{B}_{\mathbf{i}} = \mathrm{span}_A \mathcal{B}$ coincides with $\mathcal{L}(\infty)$. And also $\mathcal{B} + q\mathcal{L}(\infty)$ satisfy the property of $\mathcal{B}(\infty)$, hence, they coincide. ∎

Rmk: (a) On the combinatorial side, to apply $\tilde{F}_i$, we first pick $\mathbf{i}$ s.t. $i_1 = i$, identify $\mathcal{B} + q\mathcal{L}$ with $\mathcal{B}_{\mathbf{i}} + q\mathcal{L}$, and define

$$\boxed{\tilde{F}_i\left(F_{\mathbf{i},\beta_1=\alpha_i}^{(a_1)} F_{\mathbf{i},\beta_2}^{(a_2)} \dots F_{\mathbf{i},\beta_N}^{(a_N)}\right) = F_{\mathbf{i},\beta_1}^{(a_1+1)} F_{\mathbf{i},\beta_2}^{(a_2)} \dots F_{\mathbf{i},\beta_N}^{(a_N)}}$$

(b) Apriori, we need also operators $\tilde{E}_i$, but those are determined by $\tilde{F}_i$, due to the crystal properties.