HOMEWORK 1 (DUE JANUARY 31)

1. (a) Verify that the bilinear map $\omega\colon W\otimes W\to\mathbb{C}$ defined via $\omega(L_n,L_m)=(n^3-n)\delta_{n,-m}$ defines a 2-cocycle on the Witt algebra W. Prove that it is not a 2-coboundary.

(b) Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a nontrivial invariant symmetric bilinear form $(\cdot,\cdot)$. Verify that the bilinear map $\omega\colon \mathfrak{g}[t,t^{-1}]\otimes\mathfrak{g}[t,t^{-1}]\to\mathbb{C}$ defined via $\omega(F,G)=\mathrm{Res}_{t=0}(dF,G)$ for $F,G\in\mathfrak{g}[t,t^{-1}]$ defines a 2-cocycle on the loop algebra $\mathfrak{g}[t,t^{-1}]$. Prove that it is not a 2-coboundary.

2. Show that the Witt algebra W is a simple Lie algebra (i.e. it has no proper 2-sided ideals). Deduce that W has no nontrivial finite dimensional representations.

3. For $\alpha,\beta\in\mathbb{C}$, let $V_{\alpha,\beta}$ be the vector space of formal expressions $g(t)t^{\alpha}(dt)^{\beta}$ with $g\in\mathbb{C}[t,t^{-1}]$ (*tensor fields* of rank β and branching α on the punctured complex plane $\mathbb{C}^{\times}$).

(a) Show that the formula

$$f\partial_t\circ gt^{\alpha}(dt)^{\beta}=(fg'+\alpha t^{-1}fg+\beta f'g)t^{\alpha}(dt)^{\beta}$$

defines an action of W on $V_{\alpha,\beta}$.

(b) Choose a basis $\{v_k\}_{k\in\mathbb{Z}}$ of $V_{\alpha,\beta}$ via $v_k:=t^{k+\alpha}(dt)^{\beta}$. Verify

$$L_n(v_k)=-(k+\alpha+(n+1)\beta)v_{k+n}.$$

4. (a) Find the necessary and sufficient conditions on $(\alpha,\beta,\alpha',\beta')$ under which $V_{\alpha,\beta}$ is isomorphic to $V_{\alpha',\beta'}$.

(b) Find the necessary and sufficient conditions on (α,β) under which $V_{\alpha,\beta}$ is irreducible.

5. (a) Let $\mathfrak{a}$ be a Lie algebra, $\mathfrak{b}$ be a Lie subalgebra of $\mathfrak{a}$, M be a $\mathfrak{b}$-module, N be an $\mathfrak{a}$-module. Prove that

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M)\otimes N\simeq\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M\otimes\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{a}}(N))\ \text{as}\ \ \mathfrak{a}\text{−modules}.$$

(b) Let $\mathfrak{c}$ be a Lie algebra, $\mathfrak{a},\mathfrak{b}$ be two Lie subalgebras of $\mathfrak{c}$ such that $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}$. Note that $\mathfrak{a}\cap\mathfrak{b}$ is also a Lie subalgebra of $\mathfrak{c}$. Let M be a $\mathfrak{b}$-module. Prove that

$$\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{c}}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{c}}(M))\simeq\mathrm{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}}(\mathrm{Res}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{b}}(M))\ \text{as}\ \ \mathfrak{a}\text{−modules}.$$