HOMEWORK 2 (DUE FEBRUARY 7)

$1^\star$. (a) Complete the proof of Proposition 3(b) from Lecture 2 by proving that the corresponding $\mathcal{A}$-module $\mathrm{Diff}(x_1, x_2, \ldots)/(\mathrm{Diff}(x_1, x_2, \ldots) \cdot I_v)$ is of finite length with all composition factors isomorphic to the Fock module F_μ.

(b) Complete the proof of Theorem 1 from Lecture 3 (sketched in Lecture 4).

2. Is it true that any submodule of the Verma $\mathfrak{g}$-module M_λ is graded when $\mathfrak{g}$ is one of:

(a) the Heisenberg algebra;

(b) the Virasoro algebra;

(c) a finite-dimensional simple Lie algebra (with the *principal* grading).

3. Let $\phi\colon M_\lambda \to M_\mu$ be a nonzero homomorphism of Verma modules over a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ (with an abelian $\mathfrak{g}_0$). Prove that ϕ is injective.

4. (a) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ with a real structure $\dagger$. Define $\mathfrak{g}_\mathbb{R} := \{a \in \mathfrak{g} | a^\dagger = -a\}$. Prove that $\mathfrak{g}_\mathbb{R}$ is a Lie algebra over $\mathbb{R}$ and $\mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \simeq \mathfrak{g}$ as Lie algebras over $\mathbb{C}$.

(b) For $\lambda \in \mathfrak{g}^*_{0,\mathbb{R}}$, verify that $\overline{(M^-_{-\lambda})^{-\dagger}} \simeq M^+_\lambda$ as modules over a complex Lie algebra $\mathfrak{g}$.