HOMEWORK 3 (DUE FEBRUARY 14)

1. Let $\lambda, \mu \in \mathbb{C}$ and $\mathbf{i} := \sqrt{-1}$. Define linear operators $\{\widetilde{L}_n\}_{n\in\mathbb{Z}}$ on the Fock module F_μ via

$$\widetilde{L}_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} : a_{-j}a_{n+j} : + \mathbf{i}\lambda n a_n \text{ if } n \neq 0,$$

$$\widetilde{L}_0 = \frac{\lambda^2+\mu^2}{2} + \sum_{j>0} a_{-j}a_j.$$

Verify the following equalities in $\mathrm{End}(F_\mu)$ (this proves Proposition 1 of Lecture 5):

(a) $[\widetilde{L}_n, a_m] = -m a_{n+m} + \mathbf{i}\lambda m^2 \delta_{m,-n}\mathrm{Id}$ for all $m, n \in \mathbb{Z}$.

(b) $[\widetilde{L}_n, \widetilde{L}_m] = (n-m)\widetilde{L}_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}(1+12\lambda^2)$ for all $m, n \in \mathbb{Z}$.

Thus, we obtain an action of Vir on F_μ with the central charge $c = 1 + 12\lambda^2$.

2. Let $\delta \in \{0, 1/2\}$. Recall the algebra C_δ (generated by the fermions $\{\psi_j\}_{j\in\delta+\mathbb{Z}}$) acting on the vector space V_δ (polynomials in anticommuting variables $\{\xi_j\}_{j\in\delta+\mathbb{Z}_{\geq 0}}$) of Lecture 6. Define linear operators $\{L_n\}_{n\in\mathbb{Z}}$ on V_δ via

$$L_n = \delta_{n,0}\frac{1-2\delta}{16} + \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j : \psi_{-j}\psi_{n+j} :,$$

where the normal ordering is defined by

$$: \psi_i\psi_j := \begin{cases} \psi_i\psi_j & \text{if } i \leq j, \\ -\psi_j\psi_i & \text{if } i > j. \end{cases}$$

Verify the following equalities in $\mathrm{End}(V_\delta)$ (this proves Proposition 1 of Lecture 6):

(a) $[\psi_m, L_n] = \left(m + \frac{n}{2}\right)\psi_{m+n}$ for all $m \in \delta + \mathbb{Z}, n \in \mathbb{Z}$.

(b) $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{24}$ for all $m, n \in \mathbb{Z}$.

3. Prove that $\{\Lambda^m V, S^m V\}_{m\geq 1}$ (and more generally, the π-th Schur module $S_\pi(V) = \mathrm{Hom}_{S_n}(\pi, V^{\otimes n})$ for any $n \geq 1, \pi \in \mathrm{Irr}(S_n)$) are irreducible representations of $\mathfrak{gl}_\infty$.

4. Verify that the action of $\mathfrak{gl}_\infty$ on $\Lambda^{\frac{\infty}{2}+m}V$ constructed in the class is indeed an action (this proves Proposition 2 of Lecture 6).

5. For the Heisenberg algebra $\mathcal{A}$, consider the *quantum field* $a(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$. Let F_0 be the Fock module of $\mathcal{A}$ (with a_0 acting trivially), and let 1 denote its highest weight vector, while 1^* will denote the lowest weight vector of the dual representation. Show that

$$\langle 1^*, a(z_1)\cdots a(z_{2n})1\rangle = \sum_{\{\sigma\in S_{2n}:\sigma^2=1,\sigma(i)\neq i\ \forall i\}} \prod_{i<\sigma(i)} \frac{1}{(z_i - z_{\sigma(i)})^2},$$

where the right-hand side means the expansion in the region $|z_1| > \ldots > |z_{2n}|$.