HOMEWORK 4 (DUE FEBRUARY 21)

1. Recall the linear map $\hat{\rho}\colon \bar{\mathfrak{a}}_\infty \to \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V)$ from Lecture 7. Following our notations, we represent $A \in \bar{\mathfrak{a}}_\infty$ as the 2×2 block matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. For $A, B \in \bar{\mathfrak{a}}_\infty$, define $\alpha(A,B) \in \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V)$ via $\alpha(A,B) := [\hat{\rho}(A), \hat{\rho}(B)] - \hat{\rho}([A,B])$. Verify the following formula:

$$\alpha(A,B) = \mathrm{Tr}(A_{12}B_{21} - B_{12}A_{21}) \cdot \mathrm{Id}$$

(this proves Proposition 3 of Lecture 7).

2. For $\gamma, \beta \in \mathbb{C}$, recall the Lie algebra embedding $\overline{\varphi}_{\gamma,\beta}\colon W \hookrightarrow \bar{\mathfrak{a}}_\infty$ constructed in Lecture 7.

(a) Verify the following formula (with α as in Problem 1):

$$\alpha\left(\overline{\varphi}_{\gamma,\beta}(L_n), \overline{\varphi}_{\gamma,\beta}(L_m)\right) = \delta_{n,-m}\left(\frac{n^3-n}{12}c_\beta + 2nh_{\gamma,\beta}\right),$$

$$\text{where } c_\beta := -12\beta^2 + 12\beta - 2,\ h_{\gamma,\beta} := \frac{\gamma(\gamma+2\beta-1)}{2}.$$

(b) According to part (a) (see also Lecture 7), we get a Lie algebra embedding

$$\varphi_{\gamma,\beta}\colon \mathrm{Vir} \hookrightarrow \mathfrak{a}_\infty \text{ defined via } C \mapsto c_\beta K,\ L_n \mapsto \overline{\varphi}_{\gamma,\beta}(L_n) + \delta_{n,0}h_{\gamma,\beta}K.$$

Hence, there is a natural action of Vir on $\Lambda^{\frac{\infty}{2},m}V$ (depending on $\gamma, \beta \in \mathbb{C}$). Verify that $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \cdots \in \Lambda^{\frac{\infty}{2},m}V$ is a Vir highest weight vector of the highest weight

$$\left(\frac{(\gamma-m)(\gamma+2\beta-m-1)}{2}, -12\beta^2 + 12\beta - 2\right).$$

3. Verify the second formula from Theorem 1 of Lecture 8:

$$\Gamma^*(u) = u^{-m}z^{-1}\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right).$$

4. Let 1 (resp. ψ_0) be the highest weight vector of the bosonic space $\mathcal{B}^{(0)}$ (resp. fermionic space $\mathcal{F}^{(0)}$) and $\langle\cdot,\cdot\rangle$ be the contravariant form on that space.

(a) Compute the inner product $\langle 1, \Gamma(u_1)\cdots\Gamma(u_n)\Gamma^*(v_1)\cdots\Gamma^*(v_n)1\rangle$ by using the explicit "vertex operator" formula for $\Gamma(u), \Gamma^*(u)$.

(b) Compute analogous inner product $\langle \psi_0, X(u_1)\cdots X(u_n)X^*(v_1)\cdots X^*(v_n)\psi_0\rangle$.

(c) Equating the results of parts (a) and (b), deduce the following identity:

$$\frac{\prod_{1\le i<j\le n}(u_i-u_j)\cdot\prod_{1\le i<j\le n}(v_i-v_j)}{\prod_{i,j=1}^n(u_i-v_j)} = (-1)^{\frac{n(n-1)}{2}}\det\left(\frac{1}{u_i-v_j}\right)_{i,j=1}^n.$$

(d) Give an elementary proof of the identity from part (c).