HOMEWORK 5 (DUE FEBRUARY 28)

1. Recall the set $M(\infty) = \mathrm{Id} + \mathfrak{gl}_\infty$ and its subset $\mathrm{GL}(\infty) \subset M(\infty)$ consisting of invertible elements, introduced in Lecture 9.

(a) Show that the matrix multiplication makes $M(\infty)$ into a monoid and $\mathrm{GL}(\infty)$ into a group.

(b) Verify that the formula $A(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots) = (Av_{i_0}) \wedge (Av_{i_1}) \wedge (Av_{i_2}) \wedge \cdots$ defines an action of the monoid $M(\infty)$ and a group $\mathrm{GL}(\infty)$ on $\mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2},m}V$.

2. For $\tau \in \mathcal{F}^{(0)} \backslash \{0\}$, show that $\tau \in \Omega$ iff $S(\tau \otimes \tau) = 0$ (thus proving Theorem 5 of Lecture 9).

3. Prove that $\tau = \tau(x_1, x_2, x_3, \cdots)$ satisfies the first nontrivial equation of the KP hierarchy

$$\left(\left(\partial_{z_1}^4 + 3\partial_{z_2}^2 - 4\partial_{z_1}\partial_{z_3}\right)\tau(x-z)\tau(x+z)\right)_{|z=0} = 0$$

if and only if $u := 2\partial_x^2 \log \tau(x, y, t, c_4, c_5, \cdots)$ satisfies the KP equation

$$\frac{3}{4}\partial_y^2 u = \partial_x\left(\partial_t u - \frac{3}{2}u \cdot \partial_x u - \frac{1}{4}\partial_x^3 u\right)$$

(thus proving Proposition 1 of Lecture 10).

4. Let d be the degree operator in the Fock space $\mathcal{B}^{(0)} = F_0$ (so that d multiplies each homogeneous element by its degree, where $\deg(x_i) = i$). Let $\Gamma(u, v)$ be the operator on $\mathcal{B}^{(0)}$ introduced in Lecture 8 (see also Lecture 10). Prove the following equality of formal series:

$$\mathrm{Tr}_{\mathcal{B}^{(0)}}(\Gamma(u, v)q^d) = \prod_{n\geq 1} \frac{1-q^n}{(1-q^n u/v)(1-q^n v/u)}.$$