HOMEWORK 6 (DUE MARCH 7)

1. Prove the formula of Lecture 11 for the coefficient of the leading power of h in $\det_m(c,h)$:

$$K_m = \prod_{r,s\geq 1}^{rs\leq m} \left((2r)^s \cdot s!\right)^{m(r,s)}, \text{ where } m(r,s) = p(m-rs) - p(m-r(s+1)).$$

2. (a) Prove that $\mathcal{F} = \Lambda^{\frac{\infty}{2}} V$ is an irreducible representation of the Clifford algebra generated by $\{\hat{v}_j, \check{v}_j\}_{j\in\mathbb{Z}}$.

(b) Compute $\mathrm{Tr}_{\mathcal{F}}(q^{\mathrm{d}} z^{\mathrm{m}})$, where m is the operator multiplying elements of $\mathcal{F}^{(m)}$ by the number m, while d is the operator multiplying homogeneous elements by their degree, defined via

$$\deg(\psi_0) = 0,\ \deg(\hat{v}_j) = j,\ \deg(\check{v}_j) = -j.$$

3. (a) Using the boson-fermion isomorphism $\mathcal{F} \simeq \mathcal{B}$, compute the answer to Problem 2(b) using the bosonic realization.

(b) Deduce the Jacobi triple product identity

$$\prod_{n\geq 0}(1-q^n z)(1-q^{n+1}z^{-1})(1-q^{n+1}) = \sum_{m\in\mathbb{Z}} (-z)^m q^{\frac{m(m-1)}{2}}.$$

(c) Substitute $q = z^3$ to obtain the Euler's pentagonal identity

$$\prod_{n\geq 1}(1-z^n) = 1 + \sum_{k\geq 1}(-1)^k \left(z^{\frac{k(3k+1)}{2}} + z^{\frac{k(3k-1)}{2}}\right).$$