HOMEWORK 7 (DUE APRIL 4)

1. Prove that there exists a unique extension of the $\widehat{\mathfrak{gl}}_n$-action on $F^{(m)}$ to an action of $\widetilde{\mathfrak{gl}}_n$ with $d(\psi_m) = 0$ (thus proving Lemma 3 of Lecture 12).

2. Establish an isomorphism $\widehat{\mathfrak{gl}}_n \simeq (\widehat{\mathfrak{sl}}_n \oplus \mathcal{A})/(K_1 - K_2)$, where $K_1 = (K, 0), K_2 = (0, K)$ (thus proving Lemma 5 of Lecture 12).

3. Let $\mathfrak{g}$ be a simple Lie algebra with an invariant nondegenerate pairing $(\cdot,\cdot)$. Prove that the Casimir element (with respect to $(\cdot,\cdot)$) acts on the Verma $\mathfrak{g}$-module M_λ as $(\lambda, \lambda + 2\rho)\mathrm{Id}_{\mathrm{M}_\lambda}$.

4. Does any highest weight $\widehat{\mathfrak{g}}$-module admit a $\widetilde{\mathfrak{g}}$-action extending that of $\widehat{\mathfrak{g}}$?

5. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated to a simple Lie algebra $\mathfrak{g}$. For any $a \in \mathfrak{g}$, we set $a[n] := at^n \in \widehat{\mathfrak{g}}$ and $a(z) := \sum_{n\in\mathbb{Z}} a[n]z^{-n-1}$.

(a) Show that if V is a highest weight $\widehat{\mathfrak{g}}$-representation, then $a(z)$ defines a linear map $V \to V((z))$.

(b) Let V have a highest weight vector v with $hv = 0$ for h in the Cartan subalgebra of $\mathfrak{g}$ and $Kv = kv$ ($k \in \mathbb{C}$). Evaluate $\langle v, a(z_1)b(z_2)v\rangle$ (as a rational function).

(c) In the setup of (b), evaluate $\langle v, a(z_1)b(z_2)c(z_3)v\rangle$.