HOMEWORK 8 (DUE APRIL 11)

1. Let $A = (a_{ij})_{i,j=1}^n \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Let $\widetilde{\mathfrak{g}}(A)$ be the Lie algebra of Lecture 14, and $\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_+, \widetilde{\mathfrak{n}}_-$ be the Lie subalgebras of $\widetilde{\mathfrak{g}}(A)$ generated by $\{h_i\}_{i=1}^n, \{e_i\}_{i=1}^n, \{f_i\}_{i=1}^n$, respectively.

(a) Let $\widetilde{\mathfrak{n}}'_+$ be the free Lie algebra generated by $\{e_i\}_{i=1}^n$. Show that the universal enveloping $U(\widetilde{\mathfrak{n}}'_+)$ is a free associative algebra in $\{e_i\}_{i=1}^n$.

(b) Let $\widetilde{\mathfrak{h}}'$ be an abelian Lie algebra with a basis $\{h_i\}_{i=1}^n$. Construct an action of $\widetilde{\mathfrak{h}}'$ on $\widetilde{\mathfrak{n}}'_+$ via derivations, so that $h_i(e_j) = a_{ij}e_j$.

(c) Construct an action of $\widetilde{\mathfrak{g}}(A)$ on $U(\widetilde{\mathfrak{h}}' \ltimes \widetilde{\mathfrak{n}}'_+)$.

(d) Deduce the Lie algebra isomorphisms $\widetilde{\mathfrak{h}} \simeq \widetilde{\mathfrak{h}}'$ and $\widetilde{\mathfrak{n}}_+ \simeq \widetilde{\mathfrak{n}}'_+$.

(e) Show that the assignment $e_i \mapsto f_i, f_i \mapsto e_i, h_i \mapsto -h_i$ gives rise to a Lie algebra automorphism of $\widetilde{\mathfrak{g}}(A)$. Deduce that $\widetilde{\mathfrak{n}}_-$ is isomorphic to the free Lie algebra in $\{f_i\}_{i=1}^n$.

2. (a) Establish an isomorphism $\mathfrak{g}(A) \simeq \mathfrak{g}(A')$ if $A' = \sigma A \sigma^{-1}$ for a permutation matrix σ.

(b) Establish an isomorphism $\mathfrak{g}(A) \simeq \mathfrak{g}(A') \oplus \mathfrak{g}(A'')$ if $A = A' \oplus A''$ (that is, A has a block diagonal form with two blocks A', A'' on the diagonal).

3. (a) Compute explicitly the generalized Cartan matrices and the corresponding Dynkin diagrams for $\widehat{\mathfrak{g}}$ with $\mathfrak{g}$ being a classical simple finite dimensional Lie algebra (series $ABCD$).

(b) Compute explicitly the generalized Cartan matrices and the corresponding Dynkin diagrams for $\widehat{\mathfrak{g}}$ with $\mathfrak{g}$ being an exceptional simple finite dimensional Lie algebra (types EFG).

4. Prove that the positive part $\mathfrak{n}_+$ of the Lie algebras $\mathfrak{sl}_3, \mathfrak{sp}_4$ is generated by e_1, e_2 subject to the corresponding two Serre relations.

5. Establish (in a straightforward way) Serre relations for affinizations $\widehat{\mathfrak{g}}$ of simple finite dimensional Lie algebras $\mathfrak{g}$.

6. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$. For a $\mathfrak{g}$-representation V and $z \in \mathbb{C}^\times$, define an *evaluation representation* $V(z)$ (whose underlying vector space equals V) of $L\mathfrak{g}$ as a composition of $L\mathfrak{g} \to \mathfrak{g}, at^n \mapsto z^n \cdot a$ ($a \in \mathfrak{g}, n \in \mathbb{Z}$) and $\mathfrak{g} \to \mathrm{End}(V)$.

(a) Let $V_1, \ldots, V_n$ be irreducible nontrivial $\mathfrak{g}$-representations. Show that the $L\mathfrak{g}$-representation $V_1(z_1) \otimes \ldots \otimes V_n(z_n)$ is irreducible if and only if $z_i \neq z_j$ for $i \neq j$.

(b) When are two such irreducible representations isomorphic?

(c) Show that any irreducible finite dimensional $L\mathfrak{g}$-representation has the form as in (a).

Hint: For an irreducible finite dimensional $L\mathfrak{g}$-module V, show that $L\mathfrak{g}$-action on V factors through the action of the finite dimensional Lie algebra $\mathfrak{g} \otimes (\mathbb{C}[t, t^{-1}]/I)$, where $I \subset \mathbb{C}[t, t^{-1}]$ is an ideal of finite codimension. Apply the primary decomposition of I and Lie's theorem.