HOMEWORK 9 (DUE APRIL 18)

1. Prove that every integrable module V of finite length in category $\mathcal{O}$ over a Kac-Moody algebra is a direct sum of irreducible modules L_λ ($\lambda \in P_+$).
   Hint: Use the Casimir operator.

2. Show that Verma module M_λ over an extended Kac-Moody algebra $\mathfrak{g}_{\mathrm{ext}}(A)$ is irreducible for generic $\lambda \in P$. Specify a countable collection of hyperplanes outside of which it is true.
   Hint: Use the Casimir operator.

3. Let V be a module over an extended Kac-Moody algebra $\mathfrak{g}_{\mathrm{ext}}(A)$ from category $\mathcal{O}$. Show that for a Weil generic $\lambda \in P$ (more precisely, for λ outside of a countable collection of hyperplanes) the module $M_\lambda \otimes V$ is semisimple, and describe its decomposition into irreducibles.

4. **Vertex Operator Construction**
   Let a_i be the standard generators of the Heisenberg algebra $\mathcal{A}$. Let F_μ be the Fock representation over $\mathcal{A}$, and set $F := \oplus_{m\in\mathbb{Z}} F_{\sqrt{2}m}$. Define vertex operators on F:

$$X_\pm(u) := \exp\left(\mp\sqrt{2}\sum_{n<0}\frac{a_n}{n}u^{-n}\right)\exp\left(\mp\sqrt{2}\sum_{n>0}\frac{a_n}{n}u^{-n}\right)S^{\pm 1}u^{\pm\sqrt{2}a_0},$$

where S is the operator of shift $m \to m+1$ (cf. $\Gamma(u), \Gamma^*(u)$ of Lecture 8).

(a) Show that

$$X_a(u)X_b(v) = (u-v)^{2ab} : X_a(u)X_b(v) : \text{ for any } a, b \in \{\pm\}$$

(by abuse of notations, we identify $\pm$ with ± 1 above). In particular,

$$X_a(u)X_b(v) = X_b(v)X_a(u)$$

in the sense that the matrix elements of both sides are series in u, v which converge (but in different regions!) to the same rational functions (note that in the case of $\Gamma(u), \Gamma^*(u)$, there was a minus sign; thus, while $\Gamma(u), \Gamma^*(u)$ are "fermions", $X_+(u), X_-(u)$ are "bosons"!).

(b) Calculate $\langle 1, X_+(u_1)\cdots X_+(u_n)X_-(v_1)\cdots X_-(v_n)1\rangle$ for a highest weight vector $1 \in F_0$.

(c) Find the commutation relation between $X_\pm(u)$ and a_n.

(d) Show that the assignment

$$e(u) = \sum_{n\in\mathbb{Z}} e[n]u^{-n-1} \mapsto X_+(u),\ f(u) = \sum_{n\in\mathbb{Z}} f[n]u^{-n-1} \mapsto X_-(u),\ h[n] \mapsto \sqrt{2}a_n,\ K \mapsto \mathrm{Id}_F$$

defines an action of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ on F. Show that this is a level one highest weight representation of $\widehat{\mathfrak{sl}}_2$ with the highest weight 0 with respect to $\mathfrak{sl}_2$.

(e) Show that F is an irreducible $\widehat{\mathfrak{sl}}_2$-representation. Compute its character, i.e. $\mathrm{Tr}_F(e^{zh}q^d)$, where h is the generator of $\mathfrak{sl}_2$ and d is the degree operator defined by the conditions $d(1) = 0$ and $[d, x[n]] = nx[n]$ for any $x \in \mathfrak{sl}_2$.