HOMEWORK 10 (DUE APRIL 25)

1. Let $F := Q \otimes_{\mathbb{Z}} \mathbb{C}$ and $P := \mathfrak{h}^* \oplus F$. In Lecture 16, we constructed a particular isomorphism $\varphi\colon P \to \mathfrak{h}^*_{\mathrm{ext}}$ and introduced a symmetric nondegenerate bilinear form $(\cdot,\cdot)\colon P \times P \to \mathbb{C}$.

(a) Show that the induced pairing $(\cdot,\cdot)\colon \mathfrak{h}_{\mathrm{ext}} \times \mathfrak{h}_{\mathrm{ext}} \to \mathbb{C}$ is given by

$$(h_{\alpha_i}, h_{\alpha_j}) = d_i^{-1} a_{ij},\ (D_i, h_{\alpha_j}) = (h_{\alpha_j}, D_i) = \delta_{ij},\ (D_i, D_j) = 0.$$

(b) Extend the invariant pairing on $\mathfrak{g}(A)$ to an invariant nondegenerate pairing on $\mathfrak{g}_{\mathrm{ext}}(A)$.

2. Let $\mathfrak{g}(A) = \widehat{\mathfrak{g}}$ be the affinization of a simple finite dimensional Lie algebra $\mathfrak{g}$.

(a) Define the category $\mathcal{O}$ over $\widetilde{\mathfrak{g}} := \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$. Explain why it is basically equivalent to the category $\mathcal{O}$ over $\mathfrak{g}_{\mathrm{ext}}(A)$ as defined in Lecture 16.

(b) For a module $V \in \mathcal{O}$ of level k, verify the following relation between the actions of the Casimir operator Δ (of Lecture 17) and the Sugawara operator L_0 (of Lecture 13) on V:

$$\Delta = 2(k + h^\vee)(L_0 + d).$$

3. Let M_λ^+ (resp. M_λ^-) be the highest weight (resp. lowest weight) Verma module over a finite dimensional simple Lie algebra $\mathfrak{g}$. Let V be any $\mathfrak{h}$-diagonalizable module over $\mathfrak{g}$. Establish the isomorphism $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+ \otimes M_\mu^-, V) \simeq V[\lambda + \mu]$.

4. Let $\Theta_{n,m}(\tau, z) := \Theta_{n,m}(\tau, z, 0)$ with τ in the upper half of the complex plane.

(a) Show that for a fixed τ, this is a holomorphic function in z for all z.

(b) Relate $\Theta_{n,m}$ with $\Theta_{0,1}$.

(c) Find the zeros of $\Theta_{n,m}$ and their multiplicities.
Hint: Use Jacobi triple product to factorize $\Theta_{0,1}$.