HOMEWORK 11 (DUE MAY 2)

1. Use the explicit realization of the fundamental representation $L_d = L_{\omega_0}$ of $\widehat{\mathfrak{sl}}_2$ constructed in [Homework 9, Problem 4] to get a direct proof of the character formula (see Lecture 20):

$$\mathrm{ch}_{L_d}(h) = \Theta_{0,1}(\tau, z, u)/\varphi(q) \text{ for } h = 2\pi i(z\alpha/2 - \tau d + uK).$$

2. (a) Prove the following product formula for theta functions $\Theta_{n,m} = \Theta_{n,m}(\tau, z, u)$:

$$\Theta_{n,m} \cdot \Theta_{n',m'} = \sum_{j\in\mathbb{Z} \mod (m+m')\mathbb{Z}} d_j^{(m,m',n,n')} \Theta_{n+n'+2mj,m+m'},$$

$$d_j^{(m,m',n,n')} := \sum_{k\in \frac{m'n-mn'+2jmm'}{2mm'(m+m')}+\mathbb{Z}} q^{mm'(m+m')k^2}.$$

(b) Let $\lambda = md + \frac{n}{2}\alpha \in P_+,\ m \geq n \geq 0$. Use part (a) to prove [Lecture 20, Proposition 1]:

$$\mathrm{ch}_{L_d}(h)\mathrm{ch}_{L_\lambda}(h) = \sum_{k\in I} \psi_{m,n,k}(q)\mathrm{ch}_{L_{d+\lambda-k\alpha}}(h),$$

$$I := \left\{k \in \mathbb{Z} | -\frac{m-n+1}{2} \leq k \leq \frac{n}{2}\right\},$$

$$\psi_{m,n,k}(q) := \frac{f_k^{(m,n)}(q) - f_{n+1-k}^{(m,n)}(q)}{\varphi(q)},$$

$$f_k^{(m,n)}(q) := \sum_{j\in\mathbb{Z}} q^{(m+2)(m+3)j^2+((n+1)+2k(m+2))j+k^2}.$$

(c) For m, n, k as in part (b), define $r := n+1, s := n+1-2k$ for $k \geq 0$ and $r := m-n+1, s := m-n+2+2k$ for $k < 0$. Prove:

$$\varphi(q) \cdot q^{-k^2} \cdot \psi_{m,n,k}(q) = A + B + C, \text{ where}$$

$$A := 1 - q^{rs} - q^{(m+2-r)(m+3-s)},$$

$$B := \sum_{j>0} q^{(m+2)(m+3)j^2+((m+3)r-(m+2)s)j}\left(1 - q^{2(m+2)sj+rs}\right),$$

$$C := \sum_{j>0} q^{(m+2)(m+3)j^2-((m+3)r-(m+2)s)j}\left(1 - q^{2(m+2)(m+3-s)j+(m+2-r)(m+3-s)}\right).$$

(d) Use part (c) to provide an algebraic proof of the fact $\psi_{m,n,k}(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$.

3. Let $\mathfrak{g}$ be a simple Lie algebra with the generators $\{e_i, f_i, h_i\}_{i=1}^r$. Set $x[n] := x \cdot t^n \in L\mathfrak{g}$ for $x \in \mathfrak{g}$. Find the defining relations between elements $\{e_i[n], f_i[n], h_i[n]\}_{1\leq i\leq r}^{n\in\mathbb{Z}}$ generating $L\mathfrak{g}$.

4. Prove that for any integrable highest weight $\widehat{\mathfrak{sl}}_2$-module of level k, the current operators $e(z) := \sum_{n\in\mathbb{Z}} e[n]z^{-n-1}$ and $f(z) := \sum_{n\in\mathbb{Z}} f[n]z^{-n-1}$ satisfy $e(z)^{k+1} = f(z)^{k+1} = 0$.