HOMEWORK 12

1. (a) Prove the following inequality for a simple finite dimensional Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$:

$$(\star) \qquad \dim \operatorname{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) \leq 1 \quad \text{for any } \lambda, \mu \in \mathfrak{h}^*.$$

Hint: Prove that the Kostant partition function $P(\eta)$ is bounded (from above) by a polynomial function of η. Combine this with injectivity of any non-trivial homomorphism between Verma modules to show that for any $\eta_0 \in Q_+$, there exists $\eta \in Q_+$ satisfying $P(\eta + \eta_0) < 2P(\eta)$.

(b) Show that $(\star)$ does not always hold for Kac-Moody algebras $\mathfrak{g}(A)$.

2. Complete the proofs of Theorems 1–2 from Lecture 22 by proving the following results.

(a) Show that the leading term of $\det(\langle \cdot, \cdot \rangle^\eta)$ equals $\prod_{\alpha>0} \prod_{n \geq 1} h_\alpha^{P(\eta - n\alpha)}$, up to a nonzero constant factor.

(b) Let V be a finite dimensional space and $\{H_s\}_{s \in S}$ be a countable union of hyperplanes in V defined by linear functions $f_s \in \mathbb{C}[V]$. Let $F \in \mathbb{C}[V]$ be such that the zero set $Z(F) \subset V$ is contained in the union $\cup_{s \in S} H_s$. Show that F is a product of some linear functions f_s (possibly with multiplicities), up to a nonzero constant factor.

(c) For $\alpha \in \Delta$ with $(\alpha, \alpha) \neq 0$, establish the linear independence of the functions $\{\phi_\beta(\cdot)\}_{\beta \in \mathbb{Q}\alpha \cap Q_+}$ defined via $\phi_\beta(\eta) := P(\eta - \beta)$, $\eta \in Q_+$.

(d) Prove that any irreducible subquotient of the Verma module M_λ is of the form L_μ, and that M_λ admits a unique up to a permutation Jordan-Hölder series.