— LECTURE #1 —

This course is devoted to the study of the structure and representation theory of some of the most important infinite-dimensional Lie algebras. To be more precise, the following 3 most important examples of ∞-dim Lie algebras will be discussed:

- The Heisenberg algebra
- The Virasoro algebra ← central 1-dim extension of the Witt algebra
- The Kac-Moody algebras ← Includes finite-dim. Lie algebras & affine Lie algebras (i.e. 1-dim central extensions of $\mathfrak{g}[t,t^{-1}]$, where $\mathfrak{g}$ - simple fin. dim.)

In today's class, we will define all these algebras (assuming the field is $\mathbb{C}$)

Def 1: The Heisenberg algebra (= the oscillator algebra) $\mathcal{A}$ is equal to $\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}$ (as a vector space) with the commutator defined by

$$\boxed{[(f,\alpha),(g,\beta)] = (0, \operatorname{Res}_{t=0} g\,df) \quad \text{for } f,g\in\mathbb{C}[t,t^{-1}],\ \alpha,\beta\in\mathbb{C}}$$

Down-to-earth, $\mathcal{A}$ has a basis $\{a_n\}_{n\in\mathbb{Z}}\cup\{K\}$ with the Lie bracket:

$$\boxed{[K,a_n]=0\ ,\quad [a_n,a_m]=n\delta_{n,-m}\cdot K} \qquad \left(\begin{array}{l}\text{here } a_n=(t^n,0)\\ \phantom{\text{here }} K=(0,1)\end{array}\right)$$

Note that $\mathcal{A}$ is a 1-dim central extension of $\mathbb{C}[t,t^{-1}]$ (with the zero bracket)

Def 2: The Witt algebra is the Lie algebra of polynomial vector fields on $\mathbb{C}^\times$ (i.e. $f(t)\partial_t$, where $f\in\mathbb{C}[t,t^{-1}]$) with the natural commutator bracket:

$$\boxed{[f\partial_t, g\partial_t] = (fg'-gf')\partial_t \quad \text{for } f,g\in\mathbb{C}[t,t^{-1}]}$$

Down-to-earth, W has a basis $\{L_n\}_{n\in\mathbb{Z}}$ with the Lie bracket:

$$\boxed{[L_n,L_m]=(n-m)L_{n+m}} \qquad (\text{here } L_n=-t^{n+1}\partial_t)$$

Lemma 1: We have a natural homomorphism $\eta: W \to \mathrm{Der}\,\mathcal{A}$ defined via
$$\eta(f\partial_t)(g,\alpha) = (fg',0)$$

- First, let us verify that $\eta(f\partial_t)$ is indeed a derivation of $\mathcal{A}$, i.e.
$$\eta(f\partial_t)([(g,\alpha),(h,\beta)]) \overset{?}{=} [\eta(f\partial_t)(g,\alpha),(h,\beta)] + [(g,\alpha),\eta(f\partial_t)(h,\beta)]$$

$$\mathrm{LHS} = \eta(f\partial_t)(0, \mathrm{Res}_{t=0}\, h\,dg) = (0,0)$$

$$\mathrm{RHS} = [(fg',0),(h,\beta)] + [(g,\alpha),(fh',0)] = (0, \mathrm{Res}_{t=0}(h\,d(fg') + fh'\,dg))$$
$$= (0, \mathrm{Res}_{t=0}(hf'g' + hfg'' + fh'g')dt) = (0, \mathrm{Res}_{t=0}\, d(fg'h)) = \underset{\mathrm{LHS}}{\underset{\parallel}{(0,0)}}. \checkmark$$

- It remains to verify that η is a homomorphism:
$$[\eta(f\partial_t),\eta(g\partial_t)](h,\alpha) = (f(gh')' - g(fh')', 0) = ((fg'-gf')h', 0)$$
$$= \eta((fg'-gf')\partial_t)(h,\alpha) = \eta([f\partial_t, g\partial_t])(h,\alpha) \checkmark$$

Remark: Unlike the case of f.dim. Lie algebras, not any ∞-dim Lie alg corresponds to a Lie group. In particular, W does not correspond to any ∞-dim Lie group, but the best one can do is to consider a certain real form $W_{\mathbb{R}}$ of W whose completion corresponds to a group, which is $\mathcal{D}iff(S^1) = \{$diffeomorphisms of the unit circle$\}$. From that analogy, Lemma 1 is almost obvious. Indeed, since exponentiation provides a bijection b/w derivations of a Lie algebra and its automorphisms, the claim reduces to the fact that the residue $\mathrm{Res}_{t=0}\, g(t)df = \frac{1}{2\pi i}\oint_{S^1} g(t)\,df(t)$ is invariant under diffeom-s of S^1.

As we will see later on, a very important Lie algebra is not just W, but rather its universal central extension, known as the Virasoro algebra. We will prove next that W admits only one nontrivial central extension, thus defining Virasoro in the unique way. But for that we shall recall first the basics on central extensions of Lie algebras

Let L be a Lie algebra. Then a 1-dim central extension of L is a Lie alg $\hat{L}$ that fits into the following short exact sequence of Lie algebras:

$$0 \to \mathbb{C} \xrightarrow{\iota} \hat{L} \xrightarrow{\pi} L \to 0.$$

As a vector space $\hat{L} = L \oplus \mathbb{C}$ and the Lie bracket in $\hat{L}$ is defined via

$$\boxed{[(a,\alpha),(b,\beta)] = ([a,b], \omega(a,b))}$$

← follows from ι, π being Lie alg. homomorphisms

This formula defines a Lie bracket on $\hat{L}$ ~~iff~~:

- ω is skew-symmetric bilinear, i.e. $\boxed{\omega: \Lambda^2 L \to \mathbb{C}}$
- ω satisfies the 2-cocycle condition

(1) $$\boxed{\omega([a,b],c) + \omega([b,c],a) + \omega([c,a],b) = 0}$$

(which is equivalent to the above $[\,,]$ on $\hat{L}$ to satisfy Jacobi id)

However, this ω is determined by an original choice of splitting $\hat{L} = L \oplus \mathbb{C}$ as v.sp (i.e. a section $\varkappa: L \to \hat{L}$ of the projection $\hat{L} \xrightarrow{\pi} L$). More precisely, if ω_1, ω_2 are the two different skew-symmetric bilinear forms satisfying (1) s.t. $\hat{L}_{\omega_1}$ & $\hat{L}_{\omega_2}$ are isomorphic, then an isom. $\hat{L}_{\omega_1} \xrightarrow{\varphi} \hat{L}_{\omega_2}$ is given by $\varphi: (a,\alpha) \mapsto (a, \alpha + \xi(a))$, $\xi \in L^*$.

This map is compatible with Lie brackets iff

$$\varphi([(a,\alpha),(b,\beta)]_1) = [\varphi(a,\alpha), \varphi(b,\beta)]_2 \iff \boxed{\omega_2(a,b) - \omega_1(a,b) = \xi([a,b])}$$

Upshot: 1-dim central extensions of L are parametrized by $Z^2/B^2 =: H^2(L)$, where Z^2 is the space of 2-cocycles (i.e. skew-symm. bilinear ω satisfying (1)), B^2 is the subspace of the forms given by $(a,b) \mapsto \xi([a,b])$ called 2-coboundaries, while $H^2(L)$ is the 2nd cohomology of L.

Theorem 1: The space $H^2(W)$ is 1-dimensional, spanned by the element ω given by $\omega(L_n, L_m) = (n^3 - n)\delta_{n,-m}$.

Def 3: The Virasoro algebra Vir is the central extension of W defined by the 2-cocycle $\omega(L_n, L_m) = \frac{n^3-n}{12}\delta_{n,-m}$ (the reason why we choose exactly the factor $1/12$ will become clear later on)

Down-to-earth, Vir has a basis $\{L_n\}_{n \in \mathbb{Z}} \cup \{C\}$ with the commutator

$$\boxed{[C, L_n] = 0, \quad [L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m} \cdot C}$$

Proof of Theorem 1

Pick any $\beta \in Z^2(\overline{W})$. Consider $\xi \in W^*$ such that $\xi(L_n) = \frac{1}{n}\beta(L_n, L_0)$ for $n \neq 0$, while $\xi(L_0)$ is arbitrary, and set $\boxed{\tilde\beta(a,b) := \beta(a,b) - \xi([a,b])}$

Clearly $\tilde\beta \in Z^2(\overline{W})$, while $\tilde\beta - \beta \in B^2(\overline{W})$. The key property of $\tilde\beta$ is:

$$\tilde\beta(L_n, L_0) = \begin{cases} \tilde\beta(L_0,L_0) = 0 \text{ as } \tilde\beta\text{-skew-symm.}, & \text{if } n=0 \\ \beta(L_n,L_0) - \xi(nL_n) = 0, & \text{if } n \neq 0 \end{cases} = 0 \text{ for } \underline{\underline{\text{all } n}}.$$

• Let us now see what the 2-cocycle condition (1) gives for $a = L_0$, $b = L_m$, $c = L_n$:

$$\underbrace{\tilde\beta([L_0,L_m],L_n)}_{\substack{= -m\tilde\beta(L_m,L_n) \\ = m\cdot\tilde\beta(L_n,L_m)}} + \underbrace{\tilde\beta([L_n,L_0],L_m)}_{= n\cdot\tilde\beta(L_n,L_m)} + \underbrace{\tilde\beta([L_m,L_n],L_0)}_{= (m-n)\cdot\tilde\beta(L_{m+n},L_0) = 0 \text{ by above}} = 0$$

$$\Downarrow$$

$$(n+m)\cdot\tilde\beta(L_n,L_m) = 0$$

$\underline{\underline{\text{So}}}$: $\boxed{\tilde\beta(L_n,L_m) = \delta_{n,-m}\cdot b_n, \ b_n \in \mathbb{C}.}$ (2) in particular, $b_{-n} = -b_n$

• Let us now see what the 2-cocycle condition (1) gives for general $a = L_m$, $b = L_n$, $c = L_p$. If $m+n+p \neq 0$, then it is vacuous due to the delta-factor in (2). Hence, we may assume now that $m+n+p = 0 \Rightarrow p = -m-n$.

$$\underbrace{\tilde\beta([L_m,L_n],L_p)}_{(n-m)b_p} + \underbrace{\tilde\beta([L_n,L_p],L_m)}_{(p-n)b_m} + \underbrace{\tilde\beta([L_p,L_m],L_n)}_{(m-p)b_n} = 0$$

$$\Downarrow p = -m-n$$

$$\boxed{(n-m)b_{n+m} = (2m+n)b_n - (m+2n)b_m} \quad (3)$$

Set $\bar\beta(a,b) := \tilde\beta(a,b) - \frac{b_1}{2}\cdot\bar\xi([a,b])$, where $\bar\xi$ is the L_0-coefficient. Clearly, $\bar\beta \in Z^2(W)$ and $\bar\beta - \tilde\beta \in B^2(\overline{W})$. The key property of replacing $\tilde\beta$ with $\bar\beta$ is that we may assume $b_1 = 0$ as $\bar\beta(L_1, L_{-1}) = b_1 - \frac{b_1}{2}\bar\xi(2L_0) = 0$.

$\underline{\text{But}}$: if $b_1 = 0$, then plugging $m=1$ into (3) gives $(n-2)b_n = (n+1)b_{n-1}$, hence,

$$b_n = \frac{n+1}{n-2}\cdot\frac{n}{n-3}\cdot\frac{n-1}{n-4}\cdot\frac{n-2}{n-5}\cdot \ldots \cdot\frac{4}{1}\cdot b_2 = \frac{n^3-n}{6}b_2 \Rightarrow \boxed{b_n = \frac{n^3-n}{6}b_2} \quad (4)$$

It is straightforward to check that for any b_2, the coefficients b_n determined by (4) satisfy (3) for any m,n.

This completes our proof of Theorem 1 ■

Finally, let us define affine Kac-Moody algebras.

We shall start from a fin. dim. Lie algebra $\mathfrak{g}$ with an invariant symmetric bilinear form $(\,,\,)$. Then, the Lie algebra $\mathfrak{g}[t,t^{-1}] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$ admits the following 2-cocycle

$$\boxed{\omega(F,G) = \mathrm{Res}_{t=0}\,(G, dF) \quad \text{for any } F,G \in \mathfrak{g}[t,t^{-1}]}$$

or equivalently

$$\boxed{\omega(x\cdot f(t),\, y\cdot g(t)) = (x,y)\cdot \mathrm{Res}_{t=0}\, g\,df \quad \text{for } x,y\in\mathfrak{g},\ f,g\in\mathbb{C}[t,t^{-1}]}$$

Exercise: Check it is indeed a 2-cocycle.

Hence, it defines a 1-dim central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}[t,t^{-1}]$ with the commutator:

$$\boxed{[(F,\alpha),(G,\beta)] = ([F,G],\ \mathrm{Res}_{t=0}(G,dF))}$$

Remark: If $\mathfrak{g}=\mathbb{C}$, $(a,b)=ab$, then $\hat{\mathfrak{g}}$ coincides with the Heisenberg algebra.

Def 4: If $\mathfrak{g}$ is simple fin. dim., then $\hat{\mathfrak{g}}$ is the (untwisted) affine Kac-Moody alg.

Theorem 2: If $\mathfrak{g}$ is simple, then $H^2(\mathfrak{g}[t,t^{-1}]) = \mathbb{C}$, spanned by the aforementioned 2-cocycle ω. Thus, $\hat{\mathfrak{g}}$ is the unique nontrivial central 1-dim extension of $\mathfrak{g}[t,t^{-1}]$.

The proof will use the classical Whitehead Lemma, which we recall first.
Given a Lie algebra $\mathfrak{g}$, and a $\mathfrak{g}$-module M, define:

- $Z^1(\mathfrak{g},M) = \{\varphi : \mathfrak{g}\to M \mid \varphi([x,y]) = x\varphi(y) - y\varphi(x)\}$ ← 1-cocycles
  $\cup$
- $B^1(\mathfrak{g},M) = \{\varphi : \mathfrak{g}\to M \mid \exists\, m\in M:\ \varphi(x) = xm\ \forall x\in\mathfrak{g}\}$ ← 1-coboundaries
- $H^1(\mathfrak{g},M) := Z^1(\mathfrak{g},M)/B^1(\mathfrak{g},M)$ ← 1st cohomology of $\mathfrak{g}$ with coeff-s in M.

Whitehead Lemma: If $\mathfrak{g}$ is fin. dim. and simple, M - fin. dim., then $\boxed{H^1(\mathfrak{g},M)=0}$

We will also need the following interplay b/w $Z^2(L)$ and $Z^1(I,M)$:
let I be a Lie subalgebra of L and $M \subset L$ be a I-submodule, then $\omega\in Z^2(L)$ can be also viewed as an el-t $\varphi\in Z^1(I, M^*)$, i.e. $\varphi(x)(m) = \omega(x,m)$.

Exercise: Verify that the 2-cocycle condition for ω implies the 1-cocycle condition for φ

Pick a 2-cocycle $\beta\in Z^2(\mathfrak{g}[t,t^{-1}])$.

We shall now apply this to $L = \mathfrak{g}[t,t^{-1}]$, $\overline{L} = \mathfrak{g}\cdot t^0 \subset L$, $M = \mathfrak{g}\cdot t^n \subset L$.
Since $M \simeq \mathfrak{g}$ as a $\overline{L} \simeq \mathfrak{g}$-module, the restriction of β to $\mathfrak{g}\times\mathfrak{g}t^n$ may be viewed as an element $\varphi_n \in Z^1(\mathfrak{g},\mathfrak{g}^*)$. Due to Whitehead Lemma, $H^1(\mathfrak{g},\mathfrak{g}^*)=0$ $\Rightarrow$ $\varphi_n \in B^1(\mathfrak{g},\mathfrak{g}^*)$. Hence, there exists $m^* \in M^*$ s.t. $\varphi_n(x) = x m^*$ $\forall x \in \mathfrak{g}$.
On the other hand, by the very definition of φ_n:

$$\beta(x, y\cdot t^n) = \varphi_n(x)(yt^n) = (xm^*)(yt^n) = -m^*(x\circ(yt^n)) = -m^*([x,y]\cdot t^n) \quad \forall x,y\in\mathfrak{g}.$$

$\Rightarrow$ there is a functional $\xi_n (=-m^*) : \mathfrak{g}t^n \to \mathbb{C}$ s.t. $\boxed{\beta(x, yt^n) = \xi_n([x,y]\cdot t^n)\ \forall x,y\in\mathfrak{g}}$

Consider $\xi : \mathfrak{g}[t,t^{-1}] \to \mathbb{C}$ defined via $\xi|_{\mathfrak{g}\cdot t^n} = \xi_n$, and set
$\widetilde{\beta}(a,b) := \beta(a,b) - \xi([a,b])$ $\forall a,b \in L = \mathfrak{g}[t,t^{-1}]$. Then $\widetilde{\beta}\in Z^2(L)$, $\widetilde{\beta}-\beta \in B^2(L)$.
The key property of $\widetilde{\beta}$ is:

$$\boxed{\widetilde{\beta}(x, yt^n) = \beta(x, yt^n) - \xi_n([x,yt^n]) = 0 \quad \forall x,y\in\mathfrak{g}.}$$

- Let us apply the 2-cocycle condition (1) to $a=x$, $b=y\cdot t^n$, $c = z\cdot t^m$ $(x,y,z\in\mathfrak{g})$:

$$\widetilde{\beta}([x,y]t^n, zt^m) + \underbrace{\widetilde{\beta}([z,x]t^m, yt^n)}_{\overset{\|}{\widetilde{\beta}(yt^n,\,[x,z]t^m)}} + \underbrace{\widetilde{\beta}([y,z]t^{m+n}, x)}_{=0 \text{ by above property of } \widetilde{\beta}} = 0 \qquad \forall x,y,z\in\mathfrak{g}.$$

For fixed $n,m\in\mathbb{Z}$, this implies that $y\otimes z \mapsto \widetilde{\beta}(yt^n, zt^m)$ is an invariant form on $\mathfrak{g}$ $\Rightarrow$ $\boxed{\exists\, \lambda_{n,m}\in\mathbb{C} \text{ s.t. } \widetilde{\beta}(yt^n, zt^m) = \lambda_{n,m}\cdot(y,z)}$ Clearly: $\lambda_{n,m} = -\lambda_{m,n}$

- Let us now apply the 2-cocycle condition (1) for general $\begin{cases} a = xt^n \\ b = yt^m \\ c = zt^p \end{cases}$ $x,y,z\in\mathfrak{g}$:

$$\boxed{([x,y],z)\cdot\lambda_{n+m,p} + ([y,z],x)\cdot\lambda_{m+p,n} + ([z,x],y)\cdot\lambda_{p+n,m} = 0}$$

As $([x,y],z) = -(y,[x,z]) = ([z,x],y) = -(x,[z,y]) = ([y,z],x)$, the above is equivalent to

$$\left.\begin{array}{l}\boxed{\lambda_{n+m,p}+\lambda_{m+p,n}+\lambda_{p+n,m}=0} \\ \text{Let us write } p = N-n-m\end{array}\right\} \Rightarrow \boxed{\lambda_{n+m,\,N-n-m} = \lambda_{n,N-n} + \lambda_{m,N-m} \quad \forall\, n,m,N}$$

- set $n=m=0$ to get $\lambda_{0,N} = 2\lambda_{0,N} \Rightarrow \lambda_{0,N} = 0$ $\forall N$.
- set $m=1$ to get $\lambda_{n,N-n} + \lambda_{1,N-1} = \lambda_{n+1,N-n-1}$

$\Rightarrow \lambda_{n,N-n} = n\cdot\lambda_{1,N-1}$

- use $\lambda_{n,N-n} = -\lambda_{N-n,n}$ to get $n\cdot\lambda_{1,N-1} = -(N-n)\cdot\lambda_{1,N-1} \Rightarrow \lambda_{1,N-1} = 0$ if $N\neq 0$
$\Downarrow$
$\lambda_{m,n} = 0$ if $m+n\neq 0$.

Finally, for $m=-n$, get $\lambda_{n,-n} = n\cdot\lambda_{1,-1}$ $\Rightarrow$ get 1-dim space of solutions ∎