— LECTURE #2 —

• First, a few remarks regarding Lecture #1:

Remark 1: Applying Theorem 2 from last time to $\mathfrak{g}=\mathbb{C}$, $(\,,\,): a,b \mapsto a\cdot b$, we see that the Heisenberg algebra $\mathcal{A}$ is the unique nontrivial central extension of $\mathbb{C}[t,t^{-1}]$ (with the trivial bracket).

Remark 2: Last time we discussed what the 1-dim central extensions are, and how they are parametrized. However, it may be a bit misleading that after showing $H^2(W)$, $H^2(\mathfrak{g}[t,t^{-1}])$ are both 1-dim we concluded that Vir, $\hat{\mathfrak{g}}$ are their unique nontrivial 1-dim central extensions. The reason is that when we had a general result that 1-dim central extensions $0\to\mathbb{C}\to\hat{L}\to L\to 0$ are parametrized by $H^2(L)$, we were assuming that two such extensions $\hat{L}_1, \hat{L}_2$ are isomorphic and moreover that isomorphism fits into the following commut. diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{Id}"] & \hat{L}_1 \arrow[r] \arrow[d, "\wr"] & L \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & \hat{L}_2 \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

If however, we allow the leftmost vertical arrow to be any scalar map (i.e. we allow rescaling central element), that is, we consider central extensions $\hat{L}$ up to equivalence $\hat{L}_1 \sim \hat{L}_2$ if there is an isom. $\hat{L}_1 \xrightarrow{\sim} \hat{L}_2$ fitting into the commut. diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\lambda\cdot\mathrm{Id}\ (\lambda\neq 0)"] & \hat{L}_1 \arrow[r] \arrow[d, "\wr"] & L \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & \hat{L}_2 \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

then the equivalence classes of such nontrivial extensions are parametrized by projectivization $\mathbb{P}(H^2(L))$. In particular, if $H^2(L)$ is 1-dim, then $\mathbb{P}(H^2(L))$ contains only 1 point.

Remark 3: (a) Last time, we skipped the verifications of the 2-cocycle condition for the bilinear maps $\omega: W\otimes W\to\mathbb{C}$ and $\omega: \mathfrak{g}[t,t^{-1}]\otimes\mathfrak{g}[t,t^{-1}]\to\mathbb{C}$ used to define Vir, $\hat{\mathfrak{g}}$. This is a must-to-do excercise, see Hwk #1.

(b) Another thing that was skipped in the proofs of Thms 1, 2 last time is the verification that above $\omega$ are <u>not</u> 2-coboundaries — also in Hwk #1

- Spend ~20 minutes on finishing the proof of Theorem 2 from Lecture #1, see p. 6 of the corresponding lecture notes

- Today: A little bit of representation theory

The following is the $\infty$-dim analogue of Schur's lemma:

Lemma 1 (Dixmier's lemma): Let $V$ be a countably dimensional irreducible representation of an algebra $A$ over $\mathbb{C}$. Then any $A$-endomorphism of $V$ is a scalar, i.e. $\lambda \cdot \mathrm{Id}_V$ ($\lambda \in \mathbb{C}$). In particular, this holds when $A$ itself is of countable dimension.

▸ Consider $D = \mathrm{End}_A(V)$ (note that it is a division algebra, as $\forall \phi \in D$: $\mathrm{Ker}(\phi), \mathrm{Im}(\phi)$ are subrepresentations of $V \Rightarrow$ if $\phi \neq 0$, then $\phi$ is invertible)

First, we claim that $D$ is of countable dimension. Indeed, pick any $v \in V \setminus \{0\}$, then $V$-irred $\Rightarrow V$ is generated by $v \Rightarrow \phi \in D$ is uniquely determined by $\phi(v) \in V$, which is of countable dim.

Second, if $\phi \in D$ is not scalar, then it is transcendental $/ \mathbb{C}$ and we get $\mathbb{C}(\phi) \subset D$.

However, $\mathbb{C}(\phi)$ is uncountably dimensional, e.g. $\left\{\frac{1}{\phi - a}\right\}_{a \in \mathbb{C}}$ are lin. indep.

Hence, contradiction! ∎

Corollary 1: In the setting of Lemma 1, if $C \in A$ is central, then $C|_V$ is a scalar.

▸ Any central element defines an $A$-endomorphism of $V$ ∎

Let us now discuss some repr. theor. results of $\mathcal{A}$. Let $V$ be an irreducible $\mathcal{A}$-module. As $K \in \mathcal{A}$ is central and $U(\mathcal{A})$ is of countable dimension, we see (by Corollary 1) that $K|_V = k \cdot \mathrm{Id}_V$, $k \in \mathbb{C}$.

- Case 1: $k = 0$.

  In this case, $V$ is actually a module of the abelian Lie alg. $\mathcal{A}/\mathbb{C} \cdot K$ (or equivalently its universal enveloping algebra, which is commutative). As every element of $U(\mathcal{A}/\mathbb{C}K)$ is central, by Corollary 1, each of them acts by scalar on $V \Rightarrow V$ - 1-dim (as $V$-irred). Thus, this case is completely trivial.

- Case 2: $k \neq 0$

  Due to the existence of algebra automorphisms $\varphi_\lambda : \mathcal{A} \to \mathcal{A}$ ($\lambda \neq 0$) with $K \mapsto \lambda K$, $a_i \mapsto \begin{cases} \lambda a_i, & i > 0 \\ a_i, & i \leq 0 \end{cases}$, we may assume WLOG that $k = 1$.

Thus $V$ is a representation of $\mathcal{U}(\mathcal{A})/(K-1)$. Our next result provides an alternative description of the latter algebra.

Proposition 1: The assignment

$$\varphi: a_{-j} \mapsto x_j,\ a_j \mapsto j\frac{\partial}{\partial x_j},\ a_0 \mapsto x_0,\ K \mapsto 1 \qquad (j>0)$$

gives rise to an algebra isomorphism

$$\varphi: \mathcal{U}(\mathcal{A})/(K-1) \xrightarrow{\sim} \underbrace{\mathcal{D}iff(x_1, x_2, \ldots)}_{\text{differential operators in } x_1, x_2, \ldots \text{ with polynomial coeff-s.}} \otimes \mathbb{C}[x_0].$$

► First, the above assignment is clearly compatible with the defining relation

$$[a_n, a_m] = n\delta_{n,-m}\cdot K$$

hence, it gives rise to an algebra homom. $\varphi: \mathcal{U}(\mathcal{A})/(K-1) \to \mathcal{D}iff(x_1, x_2, \ldots) \otimes \mathbb{C}[x_0]$.

Second, it is surjective as all the generators $(x_j, \partial/\partial x_j, x_0)$ are in the image.

Finally, $\mathcal{U}(\mathcal{A})/(K-1)$ is spanned by the ordered monomials $\prod a_i^{\gamma_i}$ (by easy part of PBW) and their $\varphi$-images are clearly linearly independent $\Rightarrow$ $\varphi$-injective.

Thus, $\varphi$- isomorphism. ■

Corollary 2: For any $\mu \in \mathbb{C}$, we obtain an $\mathcal{A}$-module $F_\mu = \mathbb{C}[x_1, x_2, \ldots]$, where the diff. operators act in a natural way, while $x_0$ acts by $\mu \cdot \mathrm{Id}_V$ (note that $a_0 \in \mathcal{A}$ is also central $\Rightarrow$ $a_0$ acts by scalar on any irred. repr-n).

|| Def 1: $F_\mu$ – Fock representation.

Proposition 2: Modules $F_\mu$ are irreducible and pairwise nonisomorphic.

► As $a_0|_{F_\mu} = \mu\cdot\mathrm{Id}$ $\Rightarrow$ $F_{\mu_1} \not\simeq F_{\mu_2}$ for $\mu_1 \neq \mu_2$

To prove $F_\mu$- irreducible, assume the contradiction: Let $V \subset F_\mu$ be an $\mathcal{A}$-submodule. Pick $p(x_1, x_2, \ldots) \in V\setminus\{0\}$. Let $\underset{\neq 0}{a}\cdot x_1^{\gamma_1} x_2^{\gamma_2}\ldots$ be the monomial of the largest degree in $p$ w.r.t. lexicographic order. Then $\mathcal{D}(p) = 1$, where $\mathcal{D} = a^{-1}\cdot\left(\frac{1}{\gamma_1!}(\partial/\partial x_1)^{\gamma_1}\right)\left(\frac{1}{\gamma_2!}(\partial/\partial x_2)^{\gamma_2}\right)\cdots$ and due to Prop 1, we hence get $1 \in V$. But then $V = F_\mu$ as $F_\mu$ is generated by $1$. ■

Under suitable restrictions, $F_\mu$ are the only irreducible modules of $\mathcal{A}/(K-1)$, which is the subject of our next result.

Proposition 3: (a) Let $V$ be an irreducible $\mathcal{A}$-module in which $K=1$, $a_0=\mu$, and such that $\forall v\in V$: $\mathbb{C}[a_1,a_2,\ldots]v$ is fin. dim. and moreover all $\{a_i\}_{i>0}$ act in this space by nilpotent operators. Then: $V\simeq F_\mu$.

(b) Let $V$ be an $\mathcal{A}$-module as in (a) (but not necessarily irreducible) and such that $\forall v\in V\ \exists N\ \forall i\geq N: a_i(v)=0$. Then: $V\simeq F_\mu\otimes M$, where $M$-vector space

i.e. $V=F_\mu^{\oplus ?}$ ← direct sum of $F_\mu$.

► (a) Pick $v\in V\setminus\{0\}$ and set $W=\mathbb{C}[a_1,a_2,\ldots]v$. Then $W$ is a fin. dim. vector space with a family of pairwise commuting nilpotent operators (coming from $a_i$-action$_{i>0}$). Hence, they have a common eigenvector with eigenvalue $0$,

(Indeed, consider $W\supset W_1:=\{w\in W\mid a_1(w)=0\}$: it is non-empty and $a_i(W_1)\subset W_1\ \forall i>1$.
Next, pick $W_1\supset W_2:=\{w\in W_1\mid a_2(w)=0\}$ etc... As all $W_i$ are non-empty, we get $\bigcap W_i\neq 0$)

i.e. $\exists w\in W\setminus\{0\}$ s.t. $a_i(w)=0$ for $i>0$, $a_0(w)=\mu w$, $K(w)=w$.

But then it is easy to see (exercise!) that there is an $\mathcal{A}$-module homom. $\varphi: F_\mu\longrightarrow V$ s.t. $\varphi(1)=w$. As $F_\mu, V$-irreducible, $\varphi$-nonzero $\Rightarrow$ $\varphi$-isomorphism.

(b) Pick $v\in V\setminus\{0\}$. Let $I_v$ be the annihilator of $v$ in $\mathbb{C}[a_1,a_2,\ldots]$. Since $\mathbb{C}[a_1,a_2,\ldots]v$ is fin. dimensional, we get $\dim(\mathbb{C}[a_1,a_2,\ldots]/I_v)<\infty$. Let $W\subset V$ be the $\mathcal{A}$-submodule generated by $v$. Then $\mathcal{D}iff(x_1,x_2,\ldots)\twoheadrightarrow W$, but actually, $W$ is a quotient of

$$\mathcal{D}iff(x_1,x_2,\ldots)\Big/\big(\mathcal{D}iff(x_1,x_2,\ldots)\cdot I_v\big)=:\widetilde{W}.$$

Lemma 2: $\widetilde{W}$ is a finite-length $\mathcal{A}/(K-1, a_0-\mu)$-module with all composition factors isomorphic to $F_\mu$ (and the number of these factors equals $\dim(\mathbb{C}[a_1,a_2,\ldots]/I_v)$).

As $\widetilde{W}\twoheadrightarrow W$, Lemma 2 implies that $W$ is also a finite-length $\mathcal{A}/(K-1,a_0-\mu)$-mod with all composition factors isomorphic to $F_\mu$.

So: For any $v\in V$, the $\mathcal{A}$-submodule $U(\mathcal{A})v$ of $V$ is a finite-length module with composition factors isomorphic to $F_\mu$.

To complete the proof, we will now need the so-called Euler field

$$E=\sum_{i>0}a_{-i}a_i\in\widehat{\mathcal{A}}\text{ - certain completion of }\mathcal{A}.$$

Note that as $\forall v\in V$, we know $\exists N: a_{\geq N}(v)=0$, the action of $E$ on $V$ is well-defined.

(Continuation of the proof of Proposition 3)

Note that on the Fock space $F_\mu$, $E$ acts via $\sum_{i>0} i x_i \frac{\partial}{\partial x_i}$, hence it acts locally-finitely (i.e. $\forall v \in F_\mu$ the space $\mathbb{C}[E]v$ is fin. dim). As $\forall v \in V$, $U(\mathcal{A})v$ is of finite length with all composition factors $\simeq F_\mu$, we get:

$\boxed{\text{the action of } E \text{ on } V \text{ is locally finite.}}$

Moreover, a similar argument shows that:

$\boxed{\text{the eigenvalues of } E: V \to V \text{ are nonnegative integers}}$

(If $v \in V$ is an eigenvector of $E$, consider $W = U(\mathcal{A})v$. Then $E$ preserves $W \ni v$. But as we saw, $W$ has a composition series of finite length with all factors $\simeq F_\mu$. Finally as $E$ acts via $\sum i x_i \partial/\partial x_i$ on $F_\mu$, we see that all eigenvalues are in $\mathbb{Z}_{\geq 0}$)

$\boxed{\underline{\underline{\text{So}}}: \ V = \bigoplus_{k \geq 0} V[k], \ V[k] \text{ - generalized eigenspace of } E|_V \text{ with eigenvalue } k \in \mathbb{Z}_{\geq 0}}$

Note that if $v \in V$ is such that $a_i(v) = 0 \ \forall i > 0$, then $Ev = 0$.
Conversely, we claim that if $v \in V[0]$, then $a_i(v) = 0 \ \forall i > 0$, hence $v$ is an eigenvector of $E$ (of eigenvalue $= 0$).

[Indeed, if $\exists j > 0$ such that $a_j(v) \neq 0$, then using the commutation relation $E a_j = a_j (E - j)$, we get:
$(E+j)^s a_j(v) = a_j(E^s v) = 0$, where $s$ is chosen big enough to satisfy $E^s v = 0$.
$\Rightarrow a_j v$ is of generalized eigenvalue $\in \mathbb{Z}_{<0} \Rightarrow$ contradiction!]

$\boxed{\underline{\underline{\text{So}}}: \quad V[0] = \bigcap_{i>0} \mathrm{Ker}(a_i) = \mathrm{Ker}(E)}$

As any $v \in V[0] = \bigcap_{i>0} \mathrm{Ker}(a_i)$ determines an $\mathcal{A}$-morphism $F_\mu \to V$, $1 \mapsto v$, we may define an $\mathcal{A}$-morphism $\psi: F_\mu \otimes V[0] \longrightarrow V$ (where $\mathcal{A}$ acts only on $F_\mu$, while $V[0]$ - multiplicity vector space).
The injectivity of $\psi$ is clear (exercise!) as $\psi|_{1 \otimes V[0]}$ is injective.

Claim: $\psi$ - surjective $\Rightarrow \psi$ is an isomorphism of $\mathcal{A}$-modules

Assume not and set $\bar{V} := V/\mathrm{Im}(\psi)$. Then action of $E$ on $\bar{V}$ has strictly positive eigenvalues. On the other hand, $\forall \bar{v} \in \bar{V} \setminus 0$, the $\mathcal{A}$-submodule $\bar{W} = U(\mathcal{A})\bar{v}$ is of finite length with compos. factors $\simeq F_\mu \Rightarrow \bar{W}$ has a submodule $\simeq F_\mu$. But $1 \in F_\mu$ is of eigenvalue $0$ w.r.t. $E$, while as we just saw action of $E$ on $\bar{V}$ (hence, $\bar{W}$) has no zero eigenvalues $\Rightarrow$ Contradiction! ∎

Let us conclude today's lecture with the following observations:

- the Heisenberg algebra $\mathcal{A}$ is a $\mathbb{Z}$-graded Lie algebra, i.e. $\mathcal{A}=\bigoplus_{i\in\mathbb{Z}}\mathcal{A}_i$ with $[\mathcal{A}_i,\mathcal{A}_j]\subset\mathcal{A}_{i+j}$, where $\mathcal{A}_i=\mathbb{C}a_i\ (i\neq 0)$, $\mathcal{A}_0=\mathbb{C}a_0\oplus\mathbb{C}K$.
- the Fock representation $F_\mu$ is a $\mathbb{Z}_{\leq 0}$-graded module $/\mathcal{A}$, i.e. $F_\mu=\bigoplus_{n\geq 0}F_\mu[-n]$ with $\mathcal{A}_i(F_\mu[-n])\subset F_\mu[-n+i]$, where $F_\mu[-n]$ consists of degree $n$ polynomials in $x_1,x_2,\dots$ with $\deg(x_i)=i$

Note that $\dim F_\mu[-n]=p(n)=\#$ partitions of $n$.
Moreover, the action of the Euler field $E$ on $F_\mu$ satisfies $E_{|F_\mu[-n]}=n\cdot\mathrm{Id}$.

$$\boxed{\underline{\underline{\text{So}}}:\ \mathrm{Tr}_{F_\mu}(q^E)=\sum_{n\geq 0}\dim(F_\mu[-n])\cdot q^n=\sum_{n\geq 0}p(n)q^n=\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}}$$

<u>Sketch of the proof of Lemma 2 from our proof of Prop. 3.</u>

- For any $i>0$ $\exists m>0$ s.t. $a_i^m v=0 \Rightarrow \bar{a}_i$ (=image of $a_i$ in $\mathbb{C}[a_1,a_2,\dots]/I_v$) is nilpotent $\forall i>0$.
  Hence, the ideal $\bar{J}\subset\mathbb{C}[a_1,a_2,\dots]/I_v$ generated by $\bar{a}_i$ is gen-d by nilpotent elements.
  But $\mathbb{C}[a_1,a_2,\dots]/I_v$ is fin.dim $\Rightarrow \bar{J}$ is an ideal of the commutative ring generated by finitely many nilpotent elements.
  $\underline{\underline{\text{So}}}$: $\bar{J}$ is a nilpotent ideal $\Rightarrow \exists M\in\mathbb{Z}_{>0}: \bar{J}^M=0 \Rightarrow J^M\subset I_v$, where $J\subset\mathbb{C}[a_1,a_2,\dots]$ is the ideal gen-d by $a_1,a_2,\dots$
- A simple inductive argument shows that there is a complete flag of subspaces $\mathbb{C}[a_1,a_2,\dots]=J_0\supset J_1\supset J_2\supset\dots\supset J_{N-1}\supset J_N=I_v$ such that
  $N=\dim(\mathbb{C}[a_1,a_2,\dots]/I_v)$
  (a) $\dim(\mathbb{C}[a_1,a_2,\dots]/J_i)=i$
  (b) $J\cdot J_i\subseteq J_{i+1}$
- Finally, consider a flag $D_0\supseteq D_1\supseteq D_2\supseteq\dots\supseteq D_N$ with $D_i=\mathrm{Diff}(x_1,x_2,\_)\cdot J_i$.
  Then $\widehat{W}=D_0/D_N$ has a filtration with consequent factors $D_i/D_{i+1}$.
  It remains to notice that $D_i/D_{i+1}$ is either ZERO or $\simeq F_\mu$ (here you need $J\cdot J_i\subseteq J_{i+1}$)

This completes the proof of Lemma 2 ▪