- Spend ~15 minutes to finish the proof of Proposition 3 from last time, including
  - Lemma 2 (assigned to Oleksii last time)
  - explanation why we need this trick with the Euler vector field

- Last time we concluded with the formula

$$\mathrm{Tr}_{F_\mu}(q^E) = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

which came from the fact that $E(p(x_1,x_2,\dots)) = n\cdot p(x_1,x_2,\dots)$ $\forall\, p$ of degree n w.r.t. $\deg(x_i)=i$ and the clear equality $\sum_{n\geq 0} p(n) q^n = \frac{1}{(1-q)(1-q^2)\cdots}$

However, it is usual to shift the grading by $\frac{\mu^2}{2}$ (that will become clear latter on), so that

$$\boxed{\mathrm{ch}\, F_\mu = \sum_{n\geq 0} \dim(F_\mu[-n])\, q^{n+\frac{\mu^2}{2}} = \frac{q^{\mu^2/2}}{(1-q)(1-q^2)(1-q^3)\cdots}}$$

- Today: $\mathbb{Z}$-graded Lie algebras and their repr. theory.

Def 1: A $\mathbb{Z}$-graded Lie algebra is a Lie alg. $\mathfrak{g}$ with a grading $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, s.t.

$$\boxed{[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m} \quad \forall n,m}$$

Def 2: A $\mathbb{Z}$-graded Lie algebra is called "nondegenerate" iff:

(a) $\dim(\mathfrak{g}_n) < \infty$ $\forall n$

(b) $\mathfrak{g}_0$ – abelian

(c) $\forall n\in\mathbb{Z}_{>0}$ and a generic $\lambda\in\mathfrak{g}_0^*$, the pairing $\begin{array}{ccc} (x\,,\,y) & \longmapsto & \lambda([x,y]) \\ \mathfrak{g}_n\times\mathfrak{g}_{-n} & \longrightarrow & \mathbb{C}\end{array}$ is nondegenerate.
(i.e. λ lies in a dense open subset of $\mathfrak{g}_0^*$ w.r.t. Zariski topology)

Examples (= Exercise): (1) Heisenberg alg. $\mathcal{A}$ is $\mathbb{Z}$-graded via $\begin{cases}\mathcal{A}_n = \mathbb{C}\cdot a_n, n\neq 0 \\ \mathcal{A}_0 = \mathbb{C}a_0 \oplus \mathbb{C}K\end{cases}$, and it is nondegenerate.

(2) Witt algebra W is $\mathbb{Z}$-graded via $W_n = \mathbb{C}\cdot L_n$, and it is nondeg.

(3) Virasoro algebra Vir is $\mathbb{Z}$-graded via $\begin{cases} Vir_n = \mathbb{C}L_n, n\neq 0 \\ Vir_0 = \mathbb{C}L_0 \oplus \mathbb{C}\cdot C\end{cases}$, and it is nondeg.

(4) A simple Lie alg. $\mathfrak{g}$ is $\mathbb{Z}$-graded via $\underbrace{\deg(e_i)=1, \deg(f_i)=-1}_{\text{Chevalley generators}}$, $\deg(\overset{\text{Cartan}}{\mathfrak{h}})=0$, and it is nondegen.

(5) The affine Kac-Moody alg. $\hat{\mathfrak{g}}$ is $\mathbb{Z}$-graded via
$\deg(K)=0$, $\deg(e_i)=1$, $\deg(f_\theta\cdot t)=1$, $\deg(f_i)=-1$, $\deg(e_\theta\cdot t^{-1})=-1$, $\deg(\mathfrak{h})=0$,
where θ is the max. root of $\mathfrak{g}$. Same formulas also make $\mathfrak{g}[t,t^{-1}]$ into a $\mathbb{Z}$-graded Lie alg.
However, $\hat{\mathfrak{g}}$ is nondegenerate, while $\mathfrak{g}[t,t^{-1}]$ is not.

Set

$$\boxed{\mathfrak{n}_- := \bigoplus_{n<0}\mathfrak{g}_n, \quad \mathfrak{h} := \mathfrak{g}_0, \quad \mathfrak{n}_+ := \bigoplus_{n>0}\mathfrak{g}_n} \leftarrow \text{Lie subalgebras of } \mathfrak{g}.$$

Def 3: The triangular decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ as vector spaces

Def 4: (a) For $\lambda \in \mathfrak{h}^*$, the highest-weight Verma module $M_\lambda = M_\lambda^+$ over $\mathfrak{g}$ is defined as

$$\boxed{U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda = \mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda},$$

where $\mathbb{C}_\lambda$ denotes a 1-dim module over $\mathfrak{h}\oplus\mathfrak{n}_+$ on which $\mathfrak{n}_+$ acts by 0, while $\mathfrak{h}$ acts via λ.

(b) Likewise, the lowest-weight Verma module M_λ^- over $\mathfrak{g}$ is defined as

$$\boxed{U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \mathbb{C}_\lambda = \mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}} \mathbb{C}_\lambda}$$

where $xv=0,\ hv=\lambda(h)\cdot v\ \ \forall x\in\mathfrak{n}_-, h\in\mathfrak{h}, v\in\mathbb{C}_\lambda$

__Lemma 1__: (a) $M_\lambda^\pm$ are $\mathbb{Z}$-graded $\mathfrak{g}$-modules.

(b) $M_\lambda^+ = U(\mathfrak{n}_-)v_\lambda^+$, $M_\lambda^- = U(\mathfrak{n}_+)v_\lambda^-$ as v. spaces, where $v_\lambda^\pm \in M_\lambda^\pm$ denotes the image of $1\otimes v_\lambda^\pm$ (basis of $\mathbb{C}_\lambda$).

Recall that a module M over a graded Lie alg. $\mathfrak{g}$ is graded if $M = \bigoplus_{m\in\mathbb{Z}} M_m$, s.t.

$$\boxed{\mathfrak{g}_n(M_m)\subseteq M_{n+m}}$$

▶ (a) $\mathfrak{g}$-graded $\Rightarrow$ $U(\mathfrak{g})$-graded, $U(\mathfrak{h}\oplus\mathfrak{n}_+)$-graded

Moreover, as all el-s of $\mathfrak{n}_\pm$ kill $\mathbb{C}_\lambda$, while el-s of $\mathfrak{h}$ (which have deg$=0$) act via constants, we see that $U(\mathfrak{g})\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_\pm)}\mathbb{C}_\lambda$ is also $\mathbb{Z}$-graded.

(b) By PBW, the multiplication map $U(\mathfrak{n}_-)\otimes_{\mathbb{C}} U(\mathfrak{h}\oplus\mathfrak{n}_+) \to U(\mathfrak{g})$ is an isomorphism of vector spaces / left $U(\mathfrak{n}_-)$-modules / right $U(\mathfrak{h}\oplus\mathfrak{n}_+)$-modules.

So: $M_\lambda^+ = U(\mathfrak{g})\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)}\mathbb{C}_\lambda \simeq (U(\mathfrak{n}_-)\otimes_{\mathbb{C}} U(\mathfrak{h}\oplus\mathfrak{n}_+))\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)}\mathbb{C}_\lambda \simeq U(\mathfrak{n}_-)\otimes_{\mathbb{C}}\mathbb{C}_\lambda$ as graded $U(\mathfrak{n}_-)$-modules.

Likewise, $M_\lambda^- \simeq U(\mathfrak{n}_+)\otimes_{\mathbb{C}}\mathbb{C}_\lambda$ ■

__Corollary 1__: (a) M_λ^+ is $\mathbb{Z}_{\le 0}$-graded $\boxed{M_\lambda^+ = \bigoplus_{n\ge 0} M_\lambda^+[-n],\ \text{where } M_\lambda^+[-n] = U(\mathfrak{n}_-)[-n]\cdot v_\lambda^+}$, where $U(\mathfrak{n}_-)[-n]$ denotes the degree $-n$ subspace of $U(\mathfrak{n}_-)$.

Moreover, if $\dim(\mathfrak{g}_n)<\infty\ \forall n<0$, then:

(This follows from v.space isom. $U(\mathfrak{n}_-)\simeq S(\mathfrak{n}_-)$ due to PBW which preserves $\mathbb{Z}$-gradings)

$$\boxed{\sum_{n\ge 0}\dim(M_\lambda^+[-n])\cdot q^n = \frac{1}{\prod_{k>0}(1-q^k)^{\dim(\mathfrak{g}_{-k})}}}$$

(b) Likewise, M_λ^- is $\mathbb{Z}_{\ge 0}$-graded $\boxed{M_\lambda^- = \bigoplus_{n\ge 0} M_\lambda^-[n] \text{ with } M_\lambda^-[n] = U(\mathfrak{n}_+)[n]\cdot v_\lambda^-}$

Moreover, if $\dim(\mathfrak{g}_k)<\infty\ \forall k>0$, then:

$$\boxed{\sum_{n\ge 0}\dim(M_\lambda^-[n])q^n = \frac{1}{\prod_{k>0}(1-q^k)^{\dim(\mathfrak{g}_k)}}}$$

Lemma 2: In the particular case $\mathfrak{g} = \mathcal{A}$ (= oscillator algebra), we have a $\mathfrak{g}$-module isomorphism

$$\boxed{M^+_{1,\mu} \xrightarrow{\sim} F_\mu \quad \text{which sends } v^+_{1,\mu} \mapsto 1.}$$

$M^+_{1,\mu}$ — highest-weight Verma module with $(1,\mu) \in \mathfrak{h}^*$ sending $K \mapsto 1$, $a_0 \mapsto \mu$

F_μ — Fock repr-n of $\mathcal{A}$.

Excercise: prove this!

Our study of Verma modules rests on the $\mathfrak{g}$-bilinear form connecting M^+_λ & $M^-_{-\lambda}$.

Recall: (1) Given $\mathfrak{g}$-modules M, N, the bilinear form $(\,,\,): M \times N \to \mathbb{C}$ is called $\mathfrak{g}$-invariant if $(xm, n) + (m, xn) = 0$ $\forall x \in \mathfrak{g}$, $m \in M$, $n \in N$.

Alternatively, $(\,,\,)$ is $\mathfrak{g}$-invariant iff the associated linear map

$$M \otimes N \longrightarrow \mathbb{C}$$

is a $\mathfrak{g}$-module homomorphism, where $\mathbb{C}$ is viewed as a trivial $\mathfrak{g}$-mod.

(2) Given two $\mathbb{Z}$-graded vector spaces M, N, the bilinear form $M \times N \xrightarrow{(\,,\,)} \mathbb{C}$ is said to be of degree ZERO iff $(m, n) = 0$ for $m \in M_k$, $n \in N_l$ with $k + l \neq 0$.

Proposition 1: Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra, $\lambda \in \mathfrak{h}^*$. There exists a unique (up to scalar) $\mathfrak{g}$-invariant pairing $M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$. Moreover, it is of ZERO degree.

Def 5: Let $(\,,\,)_\lambda$ be such a $\mathfrak{g}$-inv. pairing $M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ satisfying $(v^+_\lambda, v^-_{-\lambda})_\lambda = 1$.

Our proof of Prop 1 will use the following simple lemma, which is the subject of Problem 5 on Homework 1:

Lemma 3: (a) Let $\mathfrak{a}$ - Lie alg, $\mathfrak{b} \subset \mathfrak{a}$ - Lie subalg, M - $\mathfrak{b}$-module, N - $\mathfrak{a}$-module. Then:

$$\boxed{\mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{b}}(M) \otimes N \underset{\mathfrak{a}\text{-mod}}{\simeq} \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{b}}(M \otimes \mathrm{Res}^{\mathfrak{a}}_{\mathfrak{b}}(N))}$$

(b) Let $\mathfrak{c}$ - Lie alg, $\mathfrak{a}$ & $\mathfrak{b}$ - Lie subalg-s of $\mathfrak{c}$ such that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$. Then $\mathfrak{a} \cap \mathfrak{b}$ - Lie subalg of $\mathfrak{c}$. Let M be a $\mathfrak{b}$-module. Then:

$$\boxed{\mathrm{Res}^{\mathfrak{c}}_{\mathfrak{a}}(\mathrm{Ind}^{\mathfrak{c}}_{\mathfrak{b}}(M)) \underset{\mathfrak{a}\text{-mod}}{\simeq} \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{a}\cap\mathfrak{b}}(\mathrm{Res}^{\mathfrak{b}}_{\mathfrak{a}\cap\mathfrak{b}}(M))}$$

Proof of Proposition 1

- $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+ \otimes M_{-\lambda}^-, \mathbb{C}) \overset{\text{Lemma 3(a)}}{=\!=\!=} \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+}(M_{-\lambda}^-)), \mathbb{C}) \overset{\text{Frobenius Reciprocity}}{=\!=\!=}$

$= \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+}(M_{-\lambda}^-), \mathbb{C}) \ominus$

By Lemma 3(b): $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+}(M_{-\lambda}^-) = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_-}(\mathbb{C}_{-\lambda})) = \mathrm{Ind}^{\mathfrak{h}\oplus\mathfrak{n}_+}_{\mathfrak{h}}(\mathbb{C}_{-\lambda})$

$\ominus\ \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes \mathrm{Ind}^{\mathfrak{h}\oplus\mathfrak{n}_+}_{\mathfrak{h}}(\mathbb{C}_{-\lambda}), \mathbb{C}) \overset{\text{Lemma 3(a)}}{=\!=\!=} \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathrm{Ind}^{\mathfrak{h}\oplus\mathfrak{n}_+}_{\mathfrak{h}}(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}), \mathbb{C}) \overset{\text{Frobenius Reciprocity}}{=\!=\!=}$

$= \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}, \mathbb{C}) \simeq \mathbb{C}$

So: $\boxed{\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+ \otimes M_{-\lambda}^-, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}}$

It is a straightforward check that this isom. sends a $\mathfrak{g}$-inv. pairing $(\,,\,)$ to $(v_\lambda^+, v_{-\lambda}^-)$.

- Let us finally prove that any such $\mathfrak{g}$-inv. pairing is of degree ZERO.

Let $S: U(\mathfrak{g}) \to U(\mathfrak{g})$ be the antiautomorphism given by $S(x) = -x$ for $x \in \mathfrak{g}$

the antipode of the Hopf alg. $U(\mathfrak{g})$

Then:

$$\boxed{(S(y)x v_\lambda^+, v_{-\lambda}^-) = (x v_\lambda^+, y v_{-\lambda}^-) = (v_\lambda^+, S(x) y v_{-\lambda}^-) \quad \forall x \in U(\mathfrak{n}_-), y \in U(\mathfrak{n}_+)}$$

Let $\deg(x) = -n$, $\deg(y) = m$. Then if $n+m>0 \Rightarrow S(y)xv_\lambda^+ = 0$ (as it is of degree >0)

$\Rightarrow (x v_\lambda^+, y v_{-\lambda}^-) = 0$.

Likewise, if $n+m<0 \Rightarrow S(x)y v_{-\lambda}^- = 0 \Rightarrow (x v_\lambda^+, y v_{-\lambda}^-) = 0$.

Hence, due to Lemma 1, $(\cdot,\cdot)$ is of degree ZERO ∎

Theorem 1 (Assuming that $\mathfrak{g}$ is nondegenerate): For any $n \geq 0$, the form $(\cdot,\cdot)_\lambda \big|_{M_\lambda^+[-n] \times M_{-\lambda}^-[n]}$ is nondegenerate for generic $\lambda \in \mathfrak{h}^*$.

As the proof of Theorem 1 is a bit technical, let us postpone it for a while, and first see what this result gives us for the study of repr. theory.

Corollary 2: For Weil-generic λ, $M_\lambda^\pm$ are irreducible.

i.e. away from a countable union of hypersurfaces

(this follows immediately from Theorem 1 above and Theorem 2 on the next page).

Let $J_\lambda^\pm$ be the kernel of $(\cdot,\cdot)_\lambda$ on $M_\lambda^\pm$. As $(\cdot,\cdot)_\lambda$ is of degree 0, it is clear that $J_\lambda^\pm \subseteq M_\lambda^\pm$ are $\mathbb{Z}$-graded submodules. Hence, we have graded $\mathfrak{g}$-modules:

$$\boxed{L_\lambda^\pm := M_\lambda^\pm / J_\lambda^\pm}$$

By definition of $J_\lambda^\pm$, it is clear that $(\cdot,\cdot)_\lambda$ gives rise to a nondegenerate $\mathfrak{g}$-invariant pairing $(\cdot,\cdot)_\lambda : L_\lambda^+ \times L_{-\lambda}^- \to \mathbb{C}$.

Theorem 2: (a) $L_\lambda^\pm$ - irreducible $\mathfrak{g}$-module

(b) $J_\lambda^\pm$ - the maximal proper graded submodule of $M_\lambda^\pm$

(c) If there exists $L \in \mathfrak{h} (= \mathfrak{g}_0)$ s.t. $\mathrm{ad}\, L|_{\mathfrak{g}_n} = n \cdot \mathrm{Id}$ $\forall n \in \mathbb{Z}$, then $J_\lambda^\pm$ - max. proper submodule of $M_\lambda^\pm$

(a) Assume L_λ^+ is not irred., i.e. $\exists\ 0 \neq V \subsetneq L_\lambda^+$ - $\mathfrak{g}$-submodule.

Pick any $v \in V \setminus \{0\}$ and write it as $v = \sum_{i=0}^{s} v_{-i}$ with $\deg(v_{-i}) = -i$, $v_{-s} \neq 0$

We may assume that v is chosen so that $s \in \mathbb{Z}_{\geq 0}$ is the minimal possible.

Then: $x(v) = 0$ $\forall x \in \mathfrak{n}_+ = \bigoplus_{j>0} \mathfrak{g}_j$

$\Downarrow$

$x(v_{-s}) = 0$ $\forall x \in \mathfrak{g}_j$ with $j > 0$.

Hence: $0 = -(x v_{-s}, w)_\lambda = (v_{-s}, xw)_\lambda$ $\quad \forall x \in \mathfrak{g}_{j>0}$ $\forall w \in L_{-\lambda}^-$.

But: The degree s component of $L_{-\lambda}^-$ is spanned by appropriate el-s xw as above $\Rightarrow v_{-s} \in \mathrm{Ker}(\cdot,\cdot)_\lambda \Rightarrow$ Contradiction! $\Rightarrow L_\lambda^+$ - irred.

The proof of irreducibility of L_λ^- is analogous.

(b) Want to show that J_λ^+ contains all other graded proper $\mathfrak{g}$-submodules.

Let V be another graded proper submodule of M_λ^+, and let $\bar{V}$ denote its image in L_λ^+. Note that as V is graded, proper, and M_λ^+ is gen-d by v_λ^+, then V lives in degrees $<0 \Rightarrow \bar{V}$ lives in degrees $<0 \Rightarrow \bar{V} \neq L_\lambda^+$.

But L_λ^+ is irreducible by (a) $\Rightarrow \bar{V} = 0 \Rightarrow V \subseteq J_\lambda^+ \Rightarrow J_\lambda^+$ - max. proper graded submod.

The proof for J_λ^- is analogous.

(c) It is clear that the action of $L \in \mathfrak{g}_0$ on M_λ^+ is diagonalizable with eigenvalues of the form $\{\lambda(L) + n \mid n \in \mathbb{Z}_{\leq 0}\}$. Given any $\mathfrak{g}$-submodule $V \subseteq M_\lambda^+$, the action of $L \curvearrowright V$ must be also diagonalizable $\Rightarrow V = \bigoplus_{n \geq 0} (V \cap M_\lambda^+[-n]) \Rightarrow V$ - graded.

Now (b) $\Rightarrow$ (c).

The proof for $J_\lambda^- \subset M_\lambda^-$ is analogous ∎