• Last time, we established that $\forall \lambda\in\mathfrak{h}^*$ there is a unique $\mathfrak{g}$-invariant pairing
$$(\cdot\,,\cdot)_\lambda:\ M_\lambda^+\times M_{-\lambda}^-\longrightarrow\mathbb{C}$$
such that $(v_\lambda^+, v_{-\lambda}^-)=1$. Moreover, it is of degree ZERO.

The following was stated without a proof:

Theorem 1: (Assuming $\mathfrak{g}$ is a nondegenerate $\mathbb{Z}$-graded Lie alg)
For any $n\geq 0$, the form $(\cdot,\cdot)_\lambda\big|_{M_\lambda^+[-n]\times M_{-\lambda}^-[n]}$ is nondegenerate for generic $\lambda\in\mathfrak{h}^*$

Sketch of the proof

▶ Recall that we identified $M_\lambda^+[-n]\simeq \mathcal{U}(\mathfrak{n}_-)[-n]$, $\quad M_{-\lambda}^-[n]\simeq\mathcal{U}(\mathfrak{n}_+)[n]$
$$a v_\lambda^+ \leftarrow\!\shortmid a \qquad\qquad b v_{-\lambda}^- \leftarrow\!\shortmid b$$

Hence, we may consider
$$\boxed{(\cdot\,,\cdot)_{\lambda;n}:\ \mathcal{U}(\mathfrak{n}_-)[-n]\times\mathcal{U}(\mathfrak{n}_+)[n]\longrightarrow\mathbb{C}\qquad (a,b)\longmapsto(av_\lambda^+, bv_{-\lambda}^-)_\lambda}$$

Recalling the antipode $S:\mathcal{U}(\mathfrak{g})\to\mathcal{U}(\mathfrak{g})$ (used in the proof of Prop 1 last time), we get
$(a,b)_{\lambda,n}=\left(S(b)a\, v_\lambda^+, v_{-\lambda}^-\right)_\lambda$, which implies that $(\cdot,\cdot)_{\lambda;n}$ is polynomial w.r.t. λ

(Indeed, $S(b)\in\mathcal{U}(\mathfrak{n}_+)$, $a\in\mathcal{U}(\mathfrak{n}_-)$, hence taking step-by-step factors of $S(b)$ to the right of factors of a, we either get $\mathfrak{n}_+$ acting trivially on v_λ^+, or acquire Cartan terms. In other words, we use that $\deg(S(b)a)=0$ and we have $\mathcal{U}(\mathfrak{g})\simeq\mathcal{U}(\mathfrak{n}_+)\otimes\mathcal{U}(\mathfrak{h})\otimes\mathcal{U}(\mathfrak{n}_-)$, hence, we project $S(b)a$ onto $\mathcal{U}(\mathfrak{h})\subseteq\mathcal{U}(\mathfrak{n}_+)\otimes\mathcal{U}(\mathfrak{h})\otimes\mathcal{U}(\mathfrak{n}_-)$ and act by the latter on v_λ^+)

As $\mathfrak{g}$ is nondegenerate, $\dim(\mathfrak{g}_n)=\dim(\mathfrak{g}_{-n})$ $\forall n\geq 0$. Picking bases of $\mathcal{U}(\mathfrak{n}_\mp)[\mp n]$, we may look at the determinant of $(\cdot,\cdot)_{\lambda;n}$. The latter is defined up to a factor of $\mathbb{C}^\times$, but that does not affect condition $\det(\cdot,\cdot)_{\lambda;n}\neq 0$.

So: It suffices to show that $\det(\cdot,\cdot)_{\lambda;n}\neq 0$ for generic λ.

Viewing $\det(\cdot,\cdot)_{\lambda;n}$, we wish to track down the leading term (w.r.t. λ).
For this, we start by considering the bilinear form
$$\boxed{(\cdot\,,\cdot)^0_{\lambda;n}:\ S(\mathfrak{n}_-)[-n]\times S(\mathfrak{n}_+)[n]\longrightarrow\mathbb{C}}$$
which arises from the restriction of the form $\bigoplus_{k\geq 0}\dfrac{\lambda([\cdot,\cdot])^{\otimes k}}{k!}:\ T(\mathfrak{n}_-)[-n]\times T(\mathfrak{n}_+)[n]\to\mathbb{C}$.

Explicitly, this form is glued from the following pairings (restricted to degrees $-n$ & n):
$$\boxed{\begin{array}{l}(\cdot,\cdot)^{0,k}_\lambda:\ S^k(\mathfrak{n}_-)\times S^k(\mathfrak{n}_+)\longrightarrow\mathbb{C}\qquad (a_1\cdots a_k,\ b_1\cdots b_k)\longmapsto\displaystyle\sum_{\sigma\in S_k}\lambda([a_1,b_{\sigma(1)}])\cdots\lambda([a_k,b_{\sigma(k)}])\\ \qquad\qquad\text{where } \lambda|_{\mathfrak{g}_i}\equiv 0 \text{ for } i\neq 0.\end{array}}$$

Exercise: (1) The pairing $(\cdot,\cdot)^{0,k}_{\lambda}$ is well-defined

(2) The pairing $(\cdot,\cdot)^{0}_{\lambda}$ is nondegenerate if $\lambda\in\mathfrak{h}^*$ is such that $\begin{array}{l}\mathfrak{n}_+\times\mathfrak{n}_-\longrightarrow\mathbb{C} \\ (a,b)\mapsto\lambda([a,b])\end{array}$ is nondeg.

We can also consider $\det(\cdot,\cdot)^{0}_{\lambda;n}$ for any choice of bases of $S(\mathfrak{n}_{\mp})[\mp n]$.

According to PBW, the graded vector spaces $U(\mathfrak{n}_{\mp})[\mp n]$ and $S(\mathfrak{n}_{\mp})[\mp n]$ are isom. Choosing the corresponding bases accordingly, the proof of Theorem 1 follows from

Proposition 1: $\det(\cdot,\cdot)^{0}_{\lambda;n}$ is the leading term of $\det(\cdot,\cdot)_{\lambda;n}$.

Sketch of the proof

Consider the Lie algebra $\mathfrak{g}[\varepsilon]:=\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon]$, and the Lie subalgebra $\tilde{\mathfrak{g}}\subseteq\mathfrak{g}[\varepsilon]$ generated by $\{\varepsilon\mathfrak{g}_k\}_{k\neq0}\cup\{\varepsilon^2\mathfrak{g}_0\}$. Note that $\tilde{\mathfrak{g}}/(\varepsilon-a)\simeq\mathfrak{g}$ $\forall a\in\mathbb{C}^\times$, while $\tilde{\mathfrak{g}}/(\varepsilon)\simeq\bar{\mathfrak{g}}$ — the Lie algebra, whose underlying vector space is $\mathfrak{g}=\underset{k}{\oplus}\mathfrak{g}_k$, while the Lie bracket $\bar{\mathfrak{g}}_i\times\bar{\mathfrak{g}}_j\to\bar{\mathfrak{g}}_{i+j}$ is $\begin{cases}\text{ZERO for } i+j\neq0\\ \text{equals } [\,,\,]:\mathfrak{g}_i\times\mathfrak{g}_{-i}\to\mathfrak{g}_0 \quad \text{if } i+j=0\end{cases}$

Alternatively, one can endow the vector space $\mathfrak{g}$ with a new Lie bracket, resulting in the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}^\varepsilon$ via $\boxed{[x,y]_\varepsilon=\varepsilon^{\delta_{n,0}+\delta_{m,0}-\delta_{n+m,0}+1}\cdot[x,y] \quad \forall x\in\mathfrak{g}_n,\, y\in\mathfrak{g}_m \ (n,m\in\mathbb{Z})}$

From this perspective, we note that the assignment $\underset{\mathfrak{g}_n}{\underset{\cap}{x}}\mapsto\varepsilon^{1+\delta_{n,0}}\cdot x$ defines a Lie alg. isom. $\mathfrak{g}^\varepsilon\xrightarrow[\varphi_\varepsilon]{\sim}\mathfrak{g}$ $\forall\varepsilon\neq0$.

[Underlying Idea: degenerate $\mathfrak{g}$ into the corresponding "generalized Heisenberg algebra"]

Choosing a basis of each $\mathfrak{g}_n$, the PBW thm provides a uniform basis of $U(\mathfrak{n}^\varepsilon_-)[-n]$, $U(\mathfrak{n}^\varepsilon_+)[n]$.
Let us use $(\cdot,\cdot)^{\mathfrak{g}^\varepsilon}_{\lambda;n}$ to denote the corresponding pairing for $\mathfrak{g}^\varepsilon$, and we represent it by an explicit matrix in the aforementioned basis.

Note that identifying $\mathfrak{g}^\varepsilon$ with $\mathfrak{g}$ as above via φ_ε, λ for $\mathfrak{g}^\varepsilon$ gets identified with λ/ε^2 for $\mathfrak{g}$.
Now it is clear that if x is a PBW monomial of $U(\mathfrak{n}_-)\underset{\text{v.space}}{\simeq}U(\mathfrak{n}^\varepsilon_-)$ of length $l(x)$,
y —"— of $U(\mathfrak{n}_+)\simeq U(\mathfrak{n}^\varepsilon_+)$ of length $l(y)$,

then $(xv^{+,\mathfrak{g}^\varepsilon}_{\lambda}, yv^{-,\mathfrak{g}^\varepsilon}_{-\lambda})_\lambda=(\varphi_\varepsilon(x)v^{+,\mathfrak{g}}_{\lambda/\varepsilon^2}, \varphi_\varepsilon(y)v^{-,\mathfrak{g}}_{-\lambda/\varepsilon^2})_{\lambda/\varepsilon^2}=\varepsilon^{l(x)+l(y)}\cdot(xv^{+,\mathfrak{g}}_{\lambda/\varepsilon^2}, yv^{-,\mathfrak{g}}_{-\lambda/\varepsilon^2})_{\lambda/\varepsilon^2}$.

Hence, $\boxed{\det(\cdot,\cdot)^{\mathfrak{g}^\varepsilon}_{\lambda,n}=\det(\cdot,\cdot)^{\mathfrak{g}^1=\mathfrak{g}}_{\lambda/\varepsilon^2,n}\cdot\varepsilon^{2\sum_{x:\deg(x)=-n}l(x)}}$ Viewing $\det(\cdot,\cdot)^{\mathfrak{g}^\varepsilon}_{\lambda,n}$ as a polynomial in λ,ε, the previous equality implies it is homogeneous in λ,ε^2 and the leading (in λ) term of $\det(\cdot,\cdot)^{\mathfrak{g}^1\simeq\mathfrak{g}}_{\lambda,n}$ coincides with $\det(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}=\det(\cdot,\cdot)^{\bar{\mathfrak{g}}}_{\lambda,n}$ which is easily seen to equal $\det(\cdot,\cdot)^{0,\mathfrak{g}}_{\lambda,n}$.

Thus: Indeed we see that $\det(\cdot,\cdot)^{0,\mathfrak{g}}_{\lambda,n}$ is the leading term of $\det(\cdot,\cdot)^{\mathfrak{g}}_{\lambda,n}$. ∎

Corollary 1: The Verma modules $M^{\pm}_{\lambda}$ are irreducible for Weil-generic $\lambda\in\mathfrak{h}^*$ (away from a countable union of hypersurfaces)

Follows by combining above Theorem with the fact that $L^{\pm}_{\lambda}=M^{\pm}_{\lambda}/\mathrm{Ker}(\cdot,\cdot)_\lambda$ are irreducible (see Theorem 2(a) from Lecture 3). ∎

Def 1: For a $\mathfrak{g}$-module V, a vector $v \in V$ is called singular vector of weight $\lambda \in \mathfrak{h}^*$ if

$$\boxed{hv = \lambda(h)\cdot v,\quad xv = 0 \quad \forall\, h \in \mathfrak{h},\ x \in \mathfrak{n}_+}$$

Let $\mathrm{Sing}_\lambda(V)$ denote the space of such vectors

Lemma 1: There is a canonical isomorphism $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, V) \xrightarrow{\sim} \mathrm{Sing}_\lambda(V)$

$$\phi \longmapsto \phi(v_\lambda^+)$$

►

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, V) = \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+} \mathbb{C}_\lambda, V) \xlongequal{\text{Frobenius reciprocity}} \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda, V) = \mathrm{Sing}_\lambda(V)$$ ■

Assuming $\mathfrak{g}$-nondegenerate (or at least satisfies the first two conditions: $\mathfrak{g}_0$-abelian, all $\mathfrak{g}_n$-f.dim).

Proposition 2: M_λ^+ is irreducible iff it does not have nonzero singular vectors in negative degrees (i.e. whose projection to $M_\lambda^+[0]$ is ZERO).

►

($\Rightarrow$) Assume M_λ^+ is irreducible. Assuming the contradiction, let $v \in M_\lambda^+$ be a singular vector in negative degrees. Let μ be the weight of v. According to Lemma 1, we have a $\mathfrak{g}$-homomorphism $M_\mu^+ \longrightarrow M_\lambda^+$ sending $v_\mu^+ \longmapsto v$.

But then: the image of M_μ^+ is a nonzero $\mathfrak{g}$-submodule of M_λ^+

Moreover, it is proper as the projection of this image onto $M_\lambda^+[0]$ is ZERO (since M_μ^+ is generated by v_μ^+ over $U(\mathfrak{n}_-)$, $\mathfrak{n}_-$ - in negative degrees, v - in negative deg)

This contradicts irreducibility of M_λ^+.

($\Leftarrow$) Assume M_λ^+ has no nonzero singular vectors in negative degrees. Assuming the contradiction, let M_λ^+ be not irreducible. Then $\exists\, v \in M_\lambda^+$ such that $U(\mathfrak{g})v \subsetneq M_\lambda^+$.

We can assume that v is homogenious, i.e. $v \in M_\lambda^+[-n]$ for some n.

Exercise: prove this (consider a decomposition $v = v_0 + v_1 + \dots + v_N$ with $v_j \in M_\lambda^+[-j]$ and prove that one of $v_j \neq 0$ also satisfies $U(\mathfrak{g})v_j \subsetneq M_\lambda^+$)

Then $U(\mathfrak{g})v$ is a proper graded submodule of $M_\lambda^+ \overset{\text{Thm 2}}{\Longrightarrow} U(\mathfrak{g})v \subset J_\lambda^+ \Rightarrow J_\lambda^+ \neq 0$.

Pick the largest d s.t. $J_\lambda^+[d] \neq 0$ (clearly $d \in \mathbb{Z}_{<0}$). Then $\mathfrak{n}_+(J_\lambda^+[d]) = 0$ (as $\underset{0}{\underset{\shortparallel}{J_\lambda^+[>d]}}$).

Finally, as $\mathfrak{g}_0 = \mathfrak{h}$ is abelian and $\mathfrak{h}(J_\lambda^+[d]) \subseteq J_\lambda^+[d]$, there is a 1-dim $\mathfrak{h}$-submodule in $J_\lambda^+[d]$.

Then a non-zero element $w \in J_\lambda^+[d]$ is clearly a singular vector, hence, a contradiction with the absence of singular vectors in negative deg. ■

Remark: It is clear from the proof that M_λ^+-irreducible iff it does not have nonzero homogenious singular vectors in neg. deg.

Remark: One can obviously define highest-weight / lowest-weight $\mathfrak{g}$-modules (as quotients of $M_\lambda^\pm$ by proper graded submod.)

In particular, if V is a highest-weight $\mathfrak{g}$-module with highest weight λ, then

$$M_\lambda^+ \twoheadrightarrow V \twoheadrightarrow L_\lambda^+$$

Likewise, if V is a lowest-weight $\mathfrak{g}$-module with lowest weight λ, then

$$M_\lambda^- \twoheadrightarrow V \twoheadrightarrow L_\lambda^-.$$

Now we can define categories $\mathcal{O}^{\pm}$ of $\mathfrak{g}$-modules:

Def 2: The objects of $\mathcal{O}^{+}$ are $\mathbb{C}$-graded $\mathfrak{g}$-modules such that

(a) all degrees lie in halfplane $\mathrm{Re}(z)<a$ and fall into finitely many arithmetic progressions with step -1

(b) $M[d]$ is finite dimensional $\forall d\in\mathbb{C}$

The morphisms in $\mathcal{O}^{+}$ are the graded $\mathfrak{g}$-module homomorphisms.

Replacing "$\mathrm{Re}(z)<a$" $\rightsquigarrow$ "$\mathrm{Re}(z)>a$", "step -1" $\rightsquigarrow$ "step $+1$", we get the notion of category $\mathcal{O}^{-}$.

As always, we shall assume $\mathfrak{g}$ is nondegenerate (or at least $\mathfrak{g}_0$-abelian, $\dim(\mathfrak{g}_n)<\infty\ \forall n$)

Example: $M_\lambda^{\pm}, L_\lambda^{\pm}\in\mathcal{O}^{\pm}\ \forall\lambda\in\mathfrak{h}^*$.

Proposition 3: $L_\lambda^{\pm}$ are the only irreducible objects in $\mathcal{O}^{\pm}$, and they are pairwise nonisomorphic.

▸ • Let $V\in\mathcal{O}^{+}$ be an irreducible module. Pick $d\in\mathbb{C}$ such that $V[d]\neq 0$ and $V[d+k]=0\ \forall k\in\mathbb{Z}_{>0}$. Then the action of $\mathfrak{n}_+$ annihilates $V[d]$: $\mathfrak{n}_+v=0\ \forall v\in V[d]$. On the other hand, $V[d]$ is finite dimensional and the abelian Lie algebra $\mathfrak{h}$ acts on $V[d]$ $\Rightarrow$ $V[d]$ has a 1-dim $\mathfrak{h}$-submodule, i.e. $\exists \underset{\neq 0}{v}\in V[d]$ such that $hv\in\mathbb{C}v\ \forall h\in\mathfrak{h}$ $\Rightarrow$ $\exists\lambda\in\mathfrak{h}^*$: $hv=\lambda(h)v\ \forall h\in\mathfrak{h}$.

So: $v\in V[d]\setminus\{0\}$ satisfies $xv=0$, $hv=\lambda(h)\cdot v$ $\forall x\in\mathfrak{n}_+, h\in\mathfrak{h}$.

Hence, due to Lemma 1, there is a $\mathfrak{g}$-homomorphism $\varphi: M_\lambda^+\longrightarrow V$ sending $v_\lambda^+\longmapsto v$.

As V-irreducible, φ must be surjective, i.e. $V\simeq M_\lambda/\mathrm{Ker}(\varphi)$. But as $\mathrm{Ker}(\varphi)$ is a proper graded submodule of M_λ^+ we must have $V\simeq L_\lambda^+$.

• Note that L_λ^+-irreducible $\Rightarrow$ L_λ^+ has a unique (up to scalar) vector v killed by $\mathfrak{n}_+$. Such a vector has weight λ. Hence $L_{\lambda_1}^+\simeq L_{\lambda_2}^+$ iff $\lambda_1=\lambda_2$ ∎

Def 3: For $M\in\mathcal{O}^+$, we define the character ch_M of M via

$$\boxed{\mathrm{ch}_M(q,x)=\sum_{d\in\mathbb{C}} q^{-d}\cdot \mathrm{Tr}_{M[d]}(e^{x}),\quad x\in\mathfrak{h}}$$

← power series in q (and x)

Lemma 2:
$$\mathrm{ch}_{M_\lambda^+}(q,x)=\frac{e^{\lambda(x)}}{\prod_{k>0}\det_{\mathfrak{g}_{-k}}\left(1-q^k e^{\mathrm{ad}(x)}\right)}$$

▸ This follows from identifying $M_\lambda^+\simeq U(\mathfrak{n}_-)\simeq S(\mathfrak{n}_-)$ as graded vector spaces and using the standard equality

$$\boxed{\sum_{n\geq 0} q^n\,\mathrm{Tr}_{S^nV}(S^nA)=\frac{1}{\det(1-qA)}\quad\text{for any linear operator } A: V \circlearrowleft}$$

∎

Important Remark: The Verma module M_λ^+ is not irreducible iff $\exists\, k>0$ and a singular vector $v\in M_\lambda^+[-k]$. Moreover, the smallest of such k is the smallest k s.t. $\det(\cdot,\cdot)_{\lambda,k}=0$.

Thus, the question of irreducibility of M_λ^+ boils down to the study of $\det(\cdot,\cdot)_{\lambda,k}$.

Example 1: Consider $\mathfrak{g}=\mathfrak{sl}_2$ graded via $\deg(e)=1$, $\deg(h)=0$, $\deg(f)=-1$. Identify $\mathfrak{h}^*\simeq\mathbb{C}$.
For $\lambda\in\mathbb{C}$, M_λ^+ has basis $\{f^n v_\lambda^+\}_{n\geq 0}$, $M_{-\lambda}^-$ has basis $\{e^n v_{-\lambda}^-\}_{n\geq 0}$, so that $M_\lambda^+[-n]=\mathbb{C}\cdot f^n v_\lambda^+$, $M_{-\lambda}^-[n]=\mathbb{C}\cdot e^n v_{-\lambda}^-$. The corresponding $(\cdot,\cdot)_{\lambda,n}$ is:

$$(f^n,e^n)_{\lambda,n}=(f^n v_\lambda^+,e^n v_{-\lambda}^-)_\lambda=(S(e^n)f^n v_\lambda^+,v_{-\lambda}^-)_\lambda=((-1)^n e^n f^n v_\lambda^+,v_{-\lambda}^-)\underset{\text{Exercise!}}{=}(-1)^n\cdot n!\cdot\lambda(\lambda-1)\ldots(\lambda-n+1)$$

> **Thus**: (1) M_λ^+ is irreducible iff $\lambda\notin\mathbb{Z}_{\geq 0}$.
> (2) If $\lambda\in\mathbb{Z}_{\geq 0}$, then $J_\lambda^+=\mathrm{Ker}(\cdot,\cdot)_\lambda=\mathrm{span}_{\mathbb{C}}\langle f^n v_\lambda^+\mid n\geq\lambda+1\rangle$
> $\Rightarrow L_\lambda^+=M_\lambda^+/J_\lambda^+$ is $(\lambda+1)$-dim irred. $\mathfrak{sl}_2$-module.

Example 2: Consider $\mathfrak{g}=Vir$ with $\mathfrak{g}_k=\begin{cases}\mathbb{C}\cdot L_k, & k\neq 0\\ \mathbb{C}\cdot L_0\oplus\mathbb{C}\cdot C, & k=0.\end{cases}$ We will encode $\lambda\in\mathfrak{h}^*$ be a pair $h:=\lambda(L_0)$ – conformal weight of λ, $c:=\lambda(C)$ – central charge of λ.
Hence, we will use $(\cdot,\cdot)_{h,c,k}$ to denote $(\cdot,\cdot)_{\lambda,k}$.

$k=1$ computation: In this case, we just need to compute a single pairing
$$(L_{-1},L_1)_{h,c,1}=(L_{-1}v_\lambda^+,L_1v_{-\lambda}^-)_\lambda=(-L_1L_{-1}v_\lambda^+,v_{-\lambda}^-)=(-L_{-1}L_1v_\lambda^+-2L_0v_\lambda^+,v_{-\lambda}^-)=-2h$$
$\Rightarrow(\cdot,\cdot)_{h,c,1}$ is nondegenerate iff $h\neq 0$.

$k=2$ computation: In this case, $\det(\cdot,\cdot)_{h,c,2}$ equals the determinant of
$$\begin{pmatrix}(L_{-1}^2v_\lambda^+,L_1^2v_{-\lambda}^-)_\lambda & (L_{-1}^2v_\lambda^+,L_2v_{-\lambda}^-)_\lambda\\ (L_{-2}v_\lambda^+,L_1^2v_{-\lambda}^-)_\lambda & (L_{-2}v_\lambda^+,L_2v_{-\lambda}^-)_\lambda\end{pmatrix}\underset{\text{Exercise!}}{=}\begin{pmatrix}8h^2+4h & 6h\\ -6h & -4h-\frac{1}{2}c\end{pmatrix}$$
$$\Rightarrow\det(\cdot,\cdot)_{h,c,2}=-\left((8h^2+4h)(4h+\tfrac{1}{2}c)-36h^2\right)=-4h\left((2h+1)(4h+\tfrac{c}{2})-9h\right)$$
So: $\det(\cdot,\cdot)_{h,c,2}$ vanishes on the union of the line $h=0$ and hyperbola $(2h+1)(4h+\frac{c}{2})=9h$.

Def 4: If $V=\bigoplus_{k\in\mathbb{C}}V[k]$ is a $\mathbb{C}$-graded vector space, then the restricted dual $V^\vee\subseteq V^*$ is defined via $V^\vee:=\bigoplus_{k\in\mathbb{C}}V[k]^*$ (one can make $V^\vee$ into a graded vector space itself via $V^\vee[k]=V[k]^*$ or $V^\vee[k]=V[-k]^*$)
Note: $V^{\vee\vee}\simeq V$ if $\dim V[k]<\infty\ \forall k$.

Lemma 3: If $V=\bigoplus_{k\in\mathbb{C}}V[k]$ is a $\mathbb{C}$-graded module / $\mathbb{Z}$-graded Lie alg. $\mathfrak{g}$, then the restricted dual $V^\vee$ is a $\mathbb{C}$-graded $\mathfrak{g}$-module with $V^\vee[k]=V[-k]^*$
Proof: Obvious.

<u>Proposition 4</u>: We have two mutually inverse antiequivalences of categories
$$\mathcal{O}^+ \xrightarrow{\vee} \mathcal{O}^- \quad \text{and} \quad \mathcal{O}^- \xrightarrow{\vee} \mathcal{O}^+$$
each defined by taking restricted duals.

▸ Obvious ■

From this perspective the $\mathfrak{g}$-invariant form $(\cdot,\cdot)_\lambda : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ may be viewed as a linear map $M^+_\lambda \to (M^-_{-\lambda})^\vee$, the kernel of which equals J^+_λ. Moreover, this map factors as
$$\boxed{M^+_\lambda \twoheadrightarrow L^+_\lambda \xrightarrow{\sim} (L^-_{-\lambda})^\vee \hookrightarrow (M^-_{-\lambda})^\vee}$$

<u>Corollary 2</u>: $M^+_\lambda \simeq (M^-_{-\lambda})^\vee$ if M^+_λ is irreducible (which is the case for Weyl-generic λ).

<u>Involutions</u>

Often, we will be in the following setup: $\mathfrak{g}$ – $\mathbb{Z}$-graded Lie algebra endowed with an $\underbrace{\text{involutive}}_{\omega^2 = Id}$ automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ such that $\boxed{\omega(\mathfrak{g}_n) \overset{\forall n}{=} \mathfrak{g}_{-n} \text{ and } \omega|_{\mathfrak{g}_0} = -Id}$

In this case, we clearly get equivalences of categories
$$\mathcal{O}^+ \xrightarrow{\omega} \mathcal{O}^- \quad \text{and} \quad \mathcal{O}^- \xrightarrow{\omega} \mathcal{O}^+ \qquad (\underline{\text{note}}: \text{reverse the grading } k \rightsquigarrow -k).$$
Composing those with $\vee$ we get an antiequivalence (we shall treat only one of the two):
$$\boxed{\begin{array}{l} \mathcal{O}^+ \xrightarrow{\vee} \mathcal{O}^- \xrightarrow{\omega} \mathcal{O}^+ \qquad M \longmapsto M^c \\ c \leftarrow \text{this shall be called the functor of } \underline{\text{contragredient module}} \end{array}}$$

<u>Observation</u>: There is a $\mathfrak{g}$-mod isomor. $M^+_\lambda \xrightarrow{\sim} (M^-_{-\lambda})^\omega$ given by $x \underset{U(\mathfrak{h} \oplus \mathfrak{n}_+)}{\otimes} v^+_\lambda \longmapsto \omega(x) \underset{U(\mathfrak{h} \oplus \mathfrak{n}_-)}{\otimes} v^-_{-\lambda}$ $\forall\, x \in U(\mathfrak{g})$.

Hence, the $\mathfrak{g}$-invariant form $(\cdot,\cdot)_\lambda : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ may be viewed as a $\underbrace{\mathfrak{g}\text{-contravariant}}_{(av,w)+(v,\omega(a)w)=0}$ form on M^+_λ: $\boxed{M^+_\lambda \times M^+_\lambda \xrightarrow{(\cdot,\cdot)} \mathbb{C}}$ such that $\boxed{(v^+_\lambda, v^+_\lambda) = 1}$

↑ <u>Shapovalov form</u>

Following the above discussion, we can view it as a linear map $M^+_\lambda \to (M^+_\lambda)^c$ factoring as
$$\boxed{M^+_\lambda \twoheadrightarrow L^+_\lambda \xrightarrow{\sim} (L^+_\lambda)^c \hookrightarrow (M^+_\lambda)^c}$$

<u>Lemma 4</u>: $(\cdot,\cdot)$ is symmetric.

▸ Follows from the fact that $(\cdot,\cdot)_\lambda$ is the unique $\mathfrak{g}$-inv. form with $(v^+_\lambda, v^-_{-\lambda})_\lambda = 1$ and the above bijection $\{\mathfrak{g}\text{-inv. forms}\} \leftrightarrow \{\text{contravariant forms}\}$. Indeed, the transpose of $(\cdot,\cdot)$ satisfies same conditions $\Rightarrow$ coincides with $(\cdot,\cdot)$ ■

Since any highest-weight module (of h.wt λ) V fits into $M^+_\lambda \twoheadrightarrow V \twoheadrightarrow L^+_\lambda$ and $J^+_\lambda = \mathrm{Ker}(\cdot,\cdot)$, we immediately get

<u>Corollary 3</u>: Any highest weight module carries a Shapovalov form.

Example 3: (1) $\mathfrak{g}=\mathcal{A}$, $\omega:\mathcal{A}\to\mathcal{A}$ $\quad a_k\mapsto -a_{-k},\ K\mapsto -K$

(2) $\mathfrak{g}=Vir$, $\omega: Vir\to Vir$ $\quad L_k\mapsto -L_{-k},\ C\mapsto -C$

(3) $\mathfrak{g}$-simple f.d., principal grading, $\omega:\mathfrak{g}\to\mathfrak{g}$ $\quad e_i\mapsto f_i,\ f_i\mapsto e_i,\ h_i\mapsto -h_i$

(4) $\mathfrak{g}\rightsquigarrow \hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K$, $\omega:\hat{\mathfrak{g}}\to\hat{\mathfrak{g}}$ $\quad \underset{\hat{\mathfrak{g}}}{at^k}\mapsto\omega(a)t^{-k},\ K\mapsto -K$

Finally, in what will follow in the next lectures, we need the notion of unitary representations.

Setup: $\mathfrak{g}$ – Lie algebra $/\mathbb{C}$, $\dagger:\mathfrak{g}\to\mathfrak{g}$ – $\underbrace{\text{antilinear antiinvolution}}_{\dagger^2=\mathrm{Id},\ (\lambda a)^\dagger=\bar{\lambda}\cdot a^\dagger (\forall\lambda\in\mathbb{C},a\in\mathfrak{g}),\ [a,b]^\dagger=-[a^\dagger,b^\dagger]}$

real structure on $\mathfrak{g}$

Exercise: Set $\mathfrak{g}_{\mathbb{R}}:=\{a\in\mathfrak{g}\mid a^\dagger=-a\}\leftarrow\mathbb{R}$-subspace of $\mathfrak{g}$. Show that $\mathfrak{g}_{\mathbb{R}}$ is a Lie alg$/\mathbb{R}$ and $\mathfrak{g}_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}\xrightarrow{\sim}\mathfrak{g}$.

Def 5: Let $\mathfrak{g}$ be a $\mathbb{C}$-Lie alg. with a real structure $\dagger$, and let V be a $\mathfrak{g}$-module.
V is called *Hermitian* if it is equipped with a nondeg. Hermitian form $(\cdot,\cdot)$ such that $(av,w)=(v,a^\dagger w)$. It is *unitary* if this form is positive definite.

We will be mostly interested in nondegenerate $\mathbb{Z}$-graded Lie alg-s $\mathfrak{g}$ over $\mathbb{C}$ with real structures $\dagger:\mathfrak{g}\to\mathfrak{g}$ that map $\mathfrak{g}_k\xrightarrow{\dagger}\mathfrak{g}_{-k}$. In particular, $\mathfrak{g}_0$ is an abelian Lie alg$/\mathbb{C}$ with a real structure $\dagger:\mathfrak{g}_0\circlearrowleft$.

Lemma 5: In the above setup, assume $\lambda\in\mathfrak{g}_0^*$ is such that $\overline{\lambda(a^\dagger)}=-\lambda(a)$ $\forall a\in(\mathfrak{g}_0)_{\mathbb{R}}$.
Then, the highest-weight Verma $\mathfrak{g}$-module M_λ^+ carries a Hermitian form $(\cdot,\cdot)$ such that $(v_\lambda^+,v_\lambda^+)=1$.

• Note that $-\dagger:\mathfrak{g}\to\mathfrak{g}$ is an antilinear $\mathbb{R}$-Lie alg. isom. $\Rightarrow -\dagger:\mathfrak{g}\to\bar{\mathfrak{g}}$ – $\mathbb{C}$-Lie alg. isom, where given a $\mathbb{C}$-vector space V, we use $\bar{V}$ to denote V with $\mathbb{C}$-v.space structure conjugate to that of V.

• Twisting $M_{-\lambda}^-$ by $-\dagger$, we get $(M_{-\lambda}^-)^{-\dagger}$ – module over $\mathbb{R}$-Lie alg. $\mathfrak{g}$ $\Rightarrow\overline{(M_{-\lambda}^-)^{-\dagger}}$ – module over $\mathbb{C}$-Lie alg. $\mathfrak{g}$.

Exercise: $\overline{(M_{-\lambda}^-)^{-\dagger}}\simeq M_\lambda^+$ as modules over $\mathbb{C}$-Lie alg. $\mathfrak{g}$

$x\otimes tv_{-\lambda}^-\longmapsto x^\dagger\otimes\bar{t}\,v_\lambda^+$ $\quad(x\in U(\mathfrak{g}),t\in\mathbb{C})$

Therefore, we may view $(\cdot,\cdot)_\lambda: M_\lambda^+\times M_{-\lambda}^-\to\mathbb{C}$ as a Hermitian form on M_λ^+, s.t. $(v_\lambda^+,v_\lambda^+)=1$. ∎

Corollary 4: Any highest weight module V with the highest weight $\lambda\in\mathfrak{g}_{0\mathbb{R}}^*=\{\lambda\mid\overline{\lambda(a^\dagger)}=-\lambda(a)\ \forall a\in\mathfrak{g}_0\}$ carries a Hermitian form, while L_λ^+ carries a nondegenerate Hermitian form.

Example 4: (1) $\mathfrak{g}=\mathcal{A}$, $\dagger:\mathcal{A}\to\mathcal{A}$ via $a_k^\dagger=a_{-k}$, $K^\dagger=K$

(2) $\mathfrak{g}=Vir$, $\dagger: Vir\to Vir$ via $L_k^\dagger=L_{-k}$, $C^\dagger=C$.

(3) $\mathfrak{g}$-simple f.d., $\dagger:\mathfrak{g}\to\mathfrak{g}$ via $e_i^\dagger=f_i$, $f_i^\dagger=e_i$, $h_i^\dagger=h_i$

(4) $\hat{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K$ via $(at^k)^\dagger=a^\dagger\cdot t^{-k}$, $K^\dagger=K$.