— LECTURE 5 —

- Spend ~15 min on Hermitian/unitary forms (see last page of previous notes).
- Today: Some important representations of Vir.

Recall that given a Lie algebra $\mathfrak{g}$ and a Lie algebra $\mathfrak{a}$ which is also endowed with a $\mathfrak{g}$-module structure $\rho: \mathfrak{g} \to \mathrm{End}(\mathfrak{a})$ whose image consists of derivations of $\mathfrak{a}$, one can define $\mathfrak{g} \ltimes \mathfrak{a}$:

Def 1: In the above setup, the semidirect product $\mathfrak{g} \ltimes \mathfrak{a}$ is the Lie algebra, whose underlying vector space is $\mathfrak{g} \oplus \mathfrak{a}$, but the Lie bracket is defined via

$$\boxed{[(x,\alpha),(y,\beta)] = ([x,y],\ [\alpha,\beta] + x(\beta) - y(\alpha)) \quad \text{for } x,y \in \mathfrak{g},\ \alpha,\beta \in \mathfrak{a}}$$

Rmk 1: The natural embeddings $\mathfrak{g} \hookrightarrow \mathfrak{g} \ltimes \mathfrak{a} \hookleftarrow \mathfrak{a}$ as well as the projection $\mathfrak{g} \ltimes \mathfrak{a} \twoheadrightarrow \mathfrak{g}$ are Lie alg. homomorphisms.

Exercise: Check that the above construction indeed endows v.space $\mathfrak{g} \oplus \mathfrak{a}$ with a Lie alg. structure.

Example 1: Recall that in Lemma 1 of Lecture 1, we constructed an action of the Witt alg. W on the Heisenberg alg. $\mathcal{A}$ by derivations, where $\rho(f\partial_t)(g,\alpha) = (fg', 0)$. Hence, we may consider a semidirect product $\boxed{W \ltimes \mathcal{A}}$.

Example 2: Considering the natural projection $\mathrm{Vir} \twoheadrightarrow W$ ($L_k \mapsto L_k$, $K \mapsto 0$), we can also form the semidirect product $\boxed{\mathrm{Vir} \ltimes \mathcal{A}.}$

Let us now recall the Fock representation F_μ of $\mathcal{A}$.

Question: (I) Do there exist linear operators $L_n : F_\mu \circlearrowleft$ $\forall n \in \mathbb{Z}$ s.t. $\underbrace{[L_n, a_m] = -m a_{n+m}}_{\text{explain this f-la!}}$ $\forall m \in \mathbb{Z}$?

(II) Do these operators also satisfy $[L_n, L_m] = (n-m) L_{n+m}$?

Lemma 1: For every $n \in \mathbb{Z}$, there is a unique (up to adding a constant) operator $L_n : F_\mu \circlearrowleft$ s.t. $[L_n, a_m] = -m a_{n+m}$ $\forall m \in \mathbb{Z}$

- First, let us check uniqueness. If L_n', L_n'' are two such operators $\Rightarrow [L_n' - L_n'', a_m] = 0$ $\forall m$. But F_μ is an irreducible repr. of $\mathcal{A}$ of countable dimension $\Rightarrow L_n' - L_n'' =$ scalar operator by Dixmier's Lemma.

- Let us now construct explicitly such L_n. We will need the following important notion:

Def 2: For $m,n \in \mathbb{Z}$, the normal ordered product $:a_m a_n: \in U(\mathcal{A})$ is defined via

$$\boxed{:a_m a_n: := \begin{cases} a_m a_n & \text{if } m \le n \\ a_n a_m & \text{if } m > n \end{cases}}$$

Rmk 2: (i) If $m+n \neq 0$, then $:a_m a_n: = a_m a_n$. But: $:a_m a_{-m}: = a_m a_{-m} + K \cdot \begin{cases} m, & m \geq 0 \\ 0, & m < 0. \end{cases}$

(ii) $:a_m a_n: = :a_n a_m:$

(iii) $[x, :a_m a_n:] = [x, a_m a_n]$ $\forall x \in U(\mathcal{A})$.

Let us now define $L_n : F_\mu \to F_\mu$ by the following formula:

$$\boxed{L_n := \frac{1}{2}\sum_{j\in\mathbb{Z}} :a_{-j}a_{j+n}:}$$

Key Property: While this definition of L_n involves an infinite sum, L_n viewed as an operator $F_\mu \to F_\mu$ is well-defined, since $\forall v \in F_\mu$ only finitely many of the terms $:a_{-j}a_{j+n}: v$ are nonzero! (ask why)

Rmk 3: (i) If $n\neq 0$, then $L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}$

(ii) If $n=0$, then $L_0 = \frac{\mu^2}{2} + \sum_{j>0} a_{-j}a_j$

(iii) Note that L_0 differs from the action of Euler field $E = \sum_{j>0} a_{-j}a_j$ by exactly $\frac{\mu^2}{2}$, which explains the remark from the beginning of Lecture 3, where we said that usually the degree is shifted by $\frac{\mu^2}{2}$.

To complete the proof of Lemma 1, it remains to verify $[L_n, a_m] = -m a_{n+m}$.

$$[L_n, a_m] = \frac{1}{2}\sum_{j\in\mathbb{Z}}[:a_{-j}a_{j+n}:, a_m] = \frac{1}{2}\sum_{j\in\mathbb{Z}}[a_{-j}a_{j+n}, a_m] = \frac{1}{2}\sum_{j\in\mathbb{Z}}([a_{-j}, a_m]a_{j+n} + a_{-j}[a_{j+n}, a_m])$$

$$= \frac{1}{2}(-mK\cdot a_{m+n} + a_{m+n}\cdot(-mK)) = -mK\cdot a_{m+n} \underset{\substack{K \text{ acts by } 1\\ \text{on } F_\mu}}{=\!=} -m\cdot a_{m+n}$$

Having proved Lemma 1 means that we gave an affirmative answer to part (I) of Question. As for the part (II), the answer is negative, but we shall see how the issue may be fixed.

Lemma 2: For all $n, m \in \mathbb{Z}$, the above operators $L_n, L_m : F_\mu \to F_\mu$ satisfy

$$\boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\cdot \mathrm{Id}_{F_\mu}}$$

Corollary 1: Therefore, the operators L_n extend the $\mathcal{A}$-action on F_μ to an action of $\mathrm{Vir} \ltimes \mathcal{A}$ on F_μ with the central element $C \in \mathrm{Vir}$ acting by Id_{F_μ}.

Rmk 4: This also motivates the choice of the 2-cocycle $\omega : (L_n, L_m) \mapsto \frac{n^3-n}{12}$ in the definition of Vir.

Exercise: Provide a straightforward proof of Lemma 2, based on Lemma 1.

In the class, let me give a more tricky, but less cumbersome proof.

- First, we claim that $[L_n, L_m] - (n-m)L_{n+m}$ commutes with all a_k:

$$[[L_n, L_m] - (n-m)L_{n+m}, a_k] = [[L_n, a_k], L_m] + [L_n, [L_m, a_k]] - (n-m)[L_{n+m}, a_k]$$
$$\underset{\text{Lemma 1}}{=} -k\cdot[a_{k+n}, L_m] - k[L_n, a_{k+m}] - (n-m)k\cdot a_{k+n+m} \overset{\text{Lemma 1}}{=} 0$$

- Hence, applying Dixmier's Lemma again, we see that $\boxed{[L_n, L_m] - (n-m)L_{n+m} \text{ is a scalar operator}}$
- Since $\deg(L_k) = k$, we see that $[L_n, L_m] - (n-m)L_{n+m}$ is a degree $n+m$ operator.

Thus, if $n+m \neq 0$ we must have $[L_n, L_m] = (n-m)L_{n+m}$ (as scalar operator has $\deg = 0$)

• It remains to compute the constants γ_n in the equality

$$\boxed{[L_n, L_{-n}] - 2n L_0 = \gamma_n \cdot \mathrm{Id}}$$

Note that the map $\begin{array}{c} W \times W \to \mathbb{C} \\ (L_n, L_m) \mapsto \gamma_n \cdot \delta_{n,-m} \end{array}$ must be a 2-cocycle on W. But $H^2(W) = \mathbb{C}\omega$ by Thm 1 of Lecture 1, $\omega(L_n, L_m) = \frac{n^3-n}{6}\delta_{n,-m}$

So: There is $c \in \mathbb{C}$ and $\xi \in W^*$, s.t. $\gamma_n \cdot \delta_{n,-m} = c \cdot \frac{n^3-n}{6}\delta_{n,-m} + \xi([L_n, L_m])$ $\forall n, m \in \mathbb{Z}$.

$$\Rightarrow \boxed{\gamma_n = c \cdot \frac{n^3-n}{6} + 2n \cdot \xi(L_0) \quad \forall n \in \mathbb{Z}}$$

• To find explicitly $c, \xi(L_0)$ we shall compute γ_1, γ_2.

∘ $L_0(\underset{F_\mu}{\underset{\cap}{1}}) = \frac{\mu^2}{2} \cdot 1$, $L_1(1) = 0 \Rightarrow [L_1, L_{-1}](1) = L_1 L_{-1}(1) = L_1(\mu x_1) = \mu^2$

$\Rightarrow \underbrace{[L_1, L_{-1}] - 2L_0}_{\gamma_1 \cdot \mathrm{Id}} : 1 \mapsto 1 \cdot (\mu^2 - 2 \cdot \frac{\mu^2}{2}) = 0 \Rightarrow \gamma_1 = 0 \Rightarrow \boxed{\xi(L_0) = 0.}$

∘ $L_2(1) = 0 \Rightarrow [L_2, L_{-2}](1) = L_2 L_{-2}(1) = L_2(\mu x_2 + \frac{x_1^2}{2}) = 2\mu^2 + \frac{1}{2} \Rightarrow \gamma_2 = 2\mu^2 + \frac{1}{2} - 4 \cdot \frac{\mu^2}{2} = \frac{1}{2} \Rightarrow \boxed{c = \frac{1}{2}}$

• Finally, plugging these constants back into the formula for γ_n, we get:

$$\boxed{\gamma_n = \frac{n^3-n}{12} \quad \forall n \in \mathbb{Z}}$$

■

The construction of Virasoro action of F_μ from Corollary 1 admits the following generalization.

Proposition 1: Let $\lambda, \mu \in \mathbb{C}$. Define linear operators $\tilde{L}_n : F_\mu \to F_\mu$ via

$$\boxed{\begin{aligned} \tilde{L}_n &:= \frac{1}{2}\sum_{j \in \mathbb{Z}} :a_{-j} a_{j+n}: + \underset{\sqrt{-1}}{i} \cdot \lambda n a_n = \frac{1}{2}\sum_{j \in \mathbb{Z}} a_{-j} a_{j+n} + i \cdot \lambda n a_n, \quad \text{if } n \neq 0 \\ \tilde{L}_0 &:= \frac{\lambda^2 + \mu^2}{2} + \sum_{j>0} a_{-j} a_j. \end{aligned}}$$

These operators define an action of Vir on F_μ with central charge $c = 1 + 12\lambda^2$.
Moreover, the following is an analogue of Lemma 1: $\boxed{[\tilde{L}_n, a_m] = -m a_{n+m} + i\lambda m^2 \delta_{m,-n}}$

Exercise (Hwk3): Prove this proposition!

Rmk 5: For $\lambda = 0$, this recovers the construction of Lemma 2.

Let us now discuss the unitarity properties of F_μ! We shall start from the following:

Lemma 3: For $\mu \in \mathbb{R}$, there is a unique Hermitian form $F_\mu \times F_\mu \xrightarrow{(\cdot,\cdot)} \mathbb{C}$ w.r.t. $\mathcal{A}$-action, s.t. $\langle 1, 1 \rangle = 1$

recall: $(av, w) = (v, a^\dagger w)$ $\forall a \in \mathcal{A}$, $v, w \in F_\mu$

Moreover, this form makes F_μ into the unitary $\mathcal{A}$-representation.

▷ It is easy to see that $(x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_r^{n_r}, x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_r^{m_r}) = \prod_{j=1}^{r} \delta_{n_j, m_j} \cdot \prod_{j=1}^{r} m_j! \cdot \prod_{j=1}^{r} j^{m_j}$ ← Exercise

So the monomial basis $\{x^{\vec{n}} := x_1^{n_1} x_2^{n_2} \cdots\}$ is orthogonal w.r.t. $(\,,\,)$, with $(x^{\vec{n}}, x^{\vec{n}}) \in \mathbb{R}_{>0}$

Hence, $(\,,\,)$ is positive definite! (Alternatively: Apply Corollary 4 from Lecture 4). ■

Corollary 2: If $\lambda, \mu \in \mathbb{R}$, then the Vir-representation on F_μ given by $\tilde{L}_n$ (Proposition 1) is unitary w.r.t. Hermitian structure on F_μ of Lemma 3.

► $\tilde{L}_n^{\,\dagger} = \frac{1}{2}\sum_{j\in\mathbb{Z}} (a_j a_{n-j})^\dagger + (i\lambda n a_n)^\dagger = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-n-j} a_j - i\lambda n a_{-n} = \tilde{L}_{-n}$

$\tilde{L}_0^{\,\dagger} = \frac{\lambda^2+\mu^2}{2} + \sum_{j>0} (a_{-j} a_j)^\dagger = \frac{\lambda^2+\mu^2}{2} + \sum_{j\in\mathbb{Z}} a_{-j} a_j = \tilde{L}_0$. ■

Proposition 2: Let V be a unitary representation in category $\mathcal{O}^+$ of a $\mathbb{Z}$-graded Lie alg. $\mathfrak{g}$. Then, V is a direct sum of irreducible representations.

► The proof is based on the following simple lemma:

Lemma 4: Let V be a highest weight unitary representation. Then V is irreducible.

► Let V be a h.wt. repr-n generated by the h.wt vector v_λ of weight λ.

Then: have $V \twoheadrightarrow L_\lambda$. Let $K := \mathrm{Ker}(V \twoheadrightarrow L_\lambda)$ – $\mathfrak{g}$-submodule of V $\rightsquigarrow K^\perp$ ← orthogonal complement w.r.t. unitary form, also a $\mathfrak{g}$-submodule.

But $K^\perp \cap K = 0 \Rightarrow K^\perp \hookrightarrow L_\lambda \Rightarrow K^\perp \simeq L_\lambda$.

As $V = K \oplus K^\perp \Rightarrow V \simeq L_\lambda \oplus K$, and as V is generated by v_λ that gets into L_λ, we get $K=0$ ■

Back to the proof of Proposition 2, pick $v_1 \in V_1$ ($:= V$) of largest degree (i.e. its real part is the largest) $\rightsquigarrow$ set $W_1 := U(\mathfrak{g}) v_1$ – $\mathfrak{g}$-submodule generated by v_1. Then W_1 is a highest weight unitary repr-n $\Rightarrow W_1 \simeq L_{\lambda_1}$ by Lemma 4 and $V \simeq L_{\lambda_1} \oplus V_2$. Apply the same procedure to V_2, to decompose it as $V_2 \simeq L_{\lambda_2} \oplus V_3$, etc...

As degrees of V belong to finitely many arithmetic progressions with step -1, and each degree is f.dim, the above procedure is exhaustive, which proves Prop 2 ■

Corollary 3: F_μ is a completely reducible Vir-representation for $\lambda, \mu \in \mathbb{R}$.

Note that $1 \in F_\mu$ is a singular vector w.r.t. Vir action of Proposition 1 with weight given by $h = \frac{\lambda^2+\mu^2}{2}$, $c = 1 + 12\lambda^2$. Hence, there is a natural Vir-homom. $M^+_{h,c} \longrightarrow F_\mu$, $v_{h,c} \longmapsto 1$.

Lemma 5: For Weil generic (μ, λ), this homom $M_{h,c} \longrightarrow F_\mu$ is an isomorphism.

► Both spaces are $\mathbb{Z}$-graded with all degree pieces of the same dimension (indeed, the degree $-n$ components of both spaces have dimension $p(n)$), while the homom. preserves the grading. Hence, it suffices to show it is injective. But as we know from Theorem 1 of Lecture 3, $M_{h,c}$ is irreducible for Weil generic $\lambda, \mu \Rightarrow$ homom. is injective. Hence, it is an isom. ■

Corollary 4: The irreducible h.wt. Vir-module $L_{h,c}$ is unitary if $c \geq 1$ and $h \geq \frac{c-1}{24}$.

The previous Corollary brings us to the following question

Question: When is the irreducible h.wt. Vir-representation $L_{h,c}$ has a unitary structure?

We start from the following:

Lemma 6: A necessary condition for $L_{h,c}$ to be unitary is $h \geq 0, c \geq 0$.

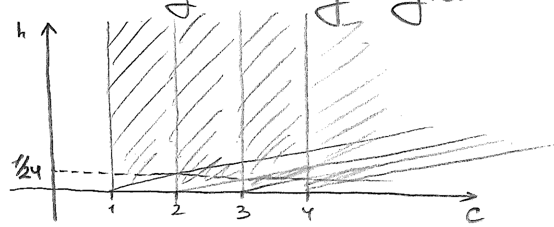
▶ We must have $\langle L_{-n} v_{\lambda}, L_{-n} v_{\lambda} \rangle \in \mathbb{R}_{\geq 0}$ $\forall n \geq 0$. $\lambda=(h,c)$

But $\langle L_{-n} v_\lambda, L_{-n} v_\lambda \rangle = \langle \omega(L_{-n}) L_{-n} v_\lambda, v_\lambda \rangle = 2n \cdot h + \frac{n^3-n}{12} \cdot c$

For $n=1$, this yields $\underline{h \in \mathbb{R}_{\geq 0}}$. Meanwhile, as n becomes large enough, the leading term is $\frac{n^3}{12} c \Rightarrow \underline{c \in \mathbb{R}_{\geq 0}}$. ■

To proceed further, note that if V_1, V_2 are two unitary $\mathfrak{g}$-representations, then $V_1 \otimes V_2$ is also unitary (the Hermitian form is $(v_1 \otimes w_1, v_2 \otimes w_2)_{V_1 \otimes V_2} = (v_1, v_2)_{V_1} \cdot (w_1, w_2)_{V_2}$). Applying this to Corollary 4, we get in particular $L_{0,1}^{\otimes(m-1)} \otimes L_{h,c}$ is unitary (given that $c \geq 1, h \geq \frac{c-1}{24}$). Considering the submodule V of $L_{0,1}^{\otimes(m-1)} \otimes L_{h,c}$ generated by $\underbrace{v_{0,1} \otimes \dots \otimes v_{0,1}}_{m-1} \otimes v_{h,c}$ (tensor product of highest wt vectors), we see that V is unitary and also h.wt $\xRightarrow{\text{Lemma 4}} V \simeq L_{h,c+m-1}$

Corollary 5: $L_{h,c}$ is unitary if $c \geq m$ & $h \geq \frac{c-m}{24}$ $(m \in \mathbb{Z}_{>0})$, i.e. $L_{h,c}$ is unitary for (h,c) in the following region:



Theorem 1: $L_{h,c}$ is unitary if $c \geq 1, h \geq 0$.

We will not prove this result at the moment.

But: Theorem 1 does not exhaust all unitary $L_{h,c}$ (e.g. $L_{0,0}$ is unitary and 1-dim). In view of Lemma 6, the question is whether there are any unitary repr. for $0 \leq c < 1$, $h \geq 0$.

We shall now construct some important unitary Vir-representations with $c = 1/2$.