— LECTURE 6 —

Def 1: Let $\delta \in \{0, \frac{1}{2}\}$. Let C_δ be the $\mathbb{C}$-algebra generated by ("fermions") $\{\psi_m \mid m \in \delta + \mathbb{Z}\}$ subject to the relations

$$\boxed{[\psi_m, \psi_n]_+ := \psi_m \psi_n + \psi_n \psi_m = \delta_{m,-n}}$$

C_0 is called the Ramond sector, $C_{1/2}$ – the Neveu-Schwarz sector

Rmk 1: C_δ is the Clifford algebra (on the vector space V with the basis $\{\psi_m\}_{m\in\delta+\mathbb{Z}}$ and the bilinear form $(\psi_m, \psi_n) = \delta_{m,-n}$)

Then, the algebra C_δ acts naturally on

$$\boxed{\begin{array}{c} V_\delta := \Lambda\{\xi_n \mid n \geq 0, n \in \delta + \mathbb{Z}\} \\ \text{where } \xi_m \xi_n = -\xi_n \xi_m \ \forall m,n \end{array}} \leftarrow \text{the space of polynomials in anticommuting variables}$$

Explicitly, $C_\delta \curvearrowright V_\delta$ via:

$$\boxed{\begin{array}{l} \psi_{-n} \xmapsto{n>0} \xi_n \leftarrow \text{left multiplication by } \xi_n \\ \psi_n \xmapsto{n>0} \partial/\partial\xi_n \leftarrow \text{given by } \partial/\partial\xi_n(\xi_{i_1} \wedge \xi_{i_2} \wedge \ldots \wedge \xi_{i_k}) = \begin{cases} 0, & \text{if } n \notin \{i_1, \ldots, i_k\} \\ (-1)^{r-1} \cdot \xi_{i_1} \wedge \ldots \wedge \xi_{i_{r-1}} \wedge \xi_{i_{r+1}} \wedge \ldots \wedge \xi_{i_k}, & \text{if } i_1 < \ldots < i_k \text{ and } n = i_r \end{cases} \\ \psi_0 \xmapsto{\text{Only for } \delta=0} \frac{1}{\sqrt{2}}\left(\xi_0 + \frac{\partial}{\partial \xi_0}\right) \end{array}}$$

Exercise: Verify that these formulas indeed define an action $C_\delta \curvearrowright V_\delta$.

Proposition 1: For $\delta \in \{0, 1/2\}$, define endomorphisms $\{L_n\}_{n\in\mathbb{Z}}$ of V_δ via

$$\boxed{L_n := \delta_{n,0} \cdot \frac{1-2\delta}{16} + \frac{1}{2} \sum_{j \in \delta + \mathbb{Z}} j :\psi_{-j} \psi_{j+n}:}$$

where $:\psi_i \psi_j: := \begin{cases} \psi_i \psi_j, & \text{if } i \leq j \\ -\psi_j \psi_i, & \text{if } i > j. \end{cases}$

Then: (1) $[\psi_m, L_n] = (m + \frac{n}{2}) \psi_{m+n}$

(2) $[L_n, L_m] = (n-m) L_{n+m} + \delta_{n,-m} \cdot \frac{m^3 - m}{24}$ $\Rightarrow$ obtain $Vir \curvearrowright V_\delta$ with $c = 1/2$.

Exercise (Hwk3): Prove this proposition!

Note that V_δ is $\mathbb{Z}/2$-graded via $\deg(\xi_i) = \bar{1} \in \mathbb{Z}/2\mathbb{Z}$, while each L_n preserves this grading. Hence, the splitting $V_\delta = \underbrace{V_\delta^+}_{\text{even el-s}} \oplus \underbrace{V_\delta^-}_{\text{odd el-s}}$ is actually a splitting of Vir-reps, i.e. $Vir : V_\delta^\pm \circlearrowleft$.

The following nontrivial result (which we will not prove) holds:

Theorem 1: For $\delta \in \{0, 1/2\}$, the actions of Vir on V_δ^+ and V_δ^- are irreducible.

- If $\delta = 0$, then $1 \in V_\delta^+$ is the highest weight vector of h.wt. $(\frac{1}{16}, \frac{1}{2})$
  $\xi_0 \in V_\delta^-$ is the highest weight vector of h.wt. $(\frac{1}{16}, \frac{1}{2})$ $\Bigg\} \xRightarrow{\text{THM1}}$ $\boxed{V_0^+ \simeq V_0^- \simeq L_{\frac{1}{16}, \frac{1}{2}}}$
- If $\delta = 1/2$, then $1 \in V_\delta^+$ is the h.wt. vector of h.wt. $(0, 1/2)$
  $\xi_{1/2} \in V_\delta^-$ is the h.wt. vector of h.wt. $(1/2, 1/2)$ $\Bigg\} \xRightarrow{\text{THM1}}$ $\boxed{\begin{array}{l} V_{1/2}^+ \simeq L_{0,1/2} \\ V_{1/2}^- \simeq L_{1/2,1/2} \end{array}}$

Finally, to relate this to the previous discussion of unitary representations, we note:

Lemma 1: Consider the Hermitian form $(\,,\,)$ on V_δ under which all monomials $\{\xi_{i_1}\wedge\xi_{i_2}\wedge\dots\wedge\xi_{i_k} \mid i_1<i_2<\dots<i_k\}$ form an orthonormal basis.

Then: (1) $(\psi_m v, w) = (v, \psi_{-m} w) \quad \forall m\in\delta+\mathbb{Z},\ v,w\in V_\delta$.

(2) $(L_n v, w) = (v, L_{-n} w) \quad \forall n\in\mathbb{Z},\ v,w\in V_\delta$

Exercise: Verify this!

Hence, V_δ and therefore $V_\delta^\pm$ are unitary.

Corollary 1: The representations $L_{0,\frac12}$, $L_{\frac1{16},\frac12}$, $L_{\frac12,\frac12}$ are unitary.

Let us conclude this discussion by computing $\underbrace{\operatorname{ch} L_{h,\frac12}}_{\sum_{\lambda\in\mathbb{C}}\dim\left(\substack{\lambda\text{-generalized}\\ \text{eigenspace}\\ L_0 \curvearrowright L_{h,1/2}}\right)\cdot q^\lambda}$ for $h=0,\frac1{16},\frac12$

$\delta=0$: $2\cdot\operatorname{ch} L_{\frac1{16},\frac12} = \operatorname{ch}(V_0^+\oplus V_0^-) = \operatorname{ch} V_0 \underset{\text{Prop 3(1)}}{=} q^{1/16}\cdot(1+1)(1+q)(1+q^2)\cdot\ldots = 2\cdot q^{1/16}\prod_{k\geq1}(1+q^k)$

So: $\boxed{\operatorname{ch} L_{1/16,1/2} = q^{1/16}\cdot\prod_{k\geq1}(1+q^k)}$

$\delta=\frac12$: $\operatorname{ch} L_{0,\frac12} + \operatorname{ch} L_{\frac12,\frac12} = \operatorname{ch}(V_0^+\oplus V_0^-) = \operatorname{ch} V_0 = (1+q^{1/2})(1+q^{3/2})(1+q^{5/2})\ldots = \prod_{k\in\frac12+\mathbb{Z}_{\geq0}}(1+q^k)$

So:
$$\boxed{\begin{aligned}\operatorname{ch} L_{0,1/2} &= \underbrace{\text{Integer part}}_{\text{taking } q^{\text{integer}}} \text{ of } \prod_{n\geq0}(1+q^{n+1/2})\\ \operatorname{ch} L_{1/2,1/2} &= \underbrace{\text{Half-integer part}}_{\text{taking } q^{1/2+\text{integer}}} \text{ of } \prod_{n\geq0}(1+q^{n+1/2})\end{aligned}}$$

The following result is beyond the scope of our class:

Theorem 2: (1) For $c = 1-\frac{6}{(n+2)(n+3)}$, $n\in\mathbb{N}$, there are finitely many h s.t. $L_{h,c}$-unitary

(2) For $0<c<1$ not of the form as in (1), there are no h s.t. $L_{h,c}$-unitary.

Let us now take a short detour, where we will discuss the previous formulas from the perspective of quantum fields (as it is usually done in physics or in vertex algebras).

Recall: If V – $\mathbb{C}$-v.space, one can associate $\mathbb{C}[z]$-modules $V[z], V[[z]], V[z,z^{-1}], V((z)), V[[z,z^{-1}]]$.

Moreover, if V is an algebra, then the first four of those are also algebras.

But: $V[[z,z^{-1}]]$ is more delicate (also note that it has torsion: $(1-z)\cdot \sum_{n\in\mathbb{Z}} vz^n=0$).

For either of the above five $\mathbb{C}[z]$-modules, one can define the linear operator

$$\boxed{\frac{d}{dz} = \text{differentiation w.r.t. } z \; : \; V[z]\circlearrowleft, \; V[[z]]\circlearrowleft, \; V[z,z^{-1}]\circlearrowleft, \; V((z))\circlearrowleft, \; V[[z,z^{-1}]]\circlearrowleft}$$

We shall also need the series, called "delta-function":

$$\boxed{\delta(w-z) := \sum_{n\in\mathbb{Z}} z^{-n-1} w^n}$$

Motivation: If $f(z)\in V[z,z^{-1}]$, then $\frac{1}{2\pi i}\oint_{|z|=1} \delta(w-z) f(z)\, dz = f(w)$.

- Let us now encode all the generators $\{a_n\}_{n\in\mathbb{Z}}$ of the Heisenberg algebra $\mathcal{A}$ into the so-called quantum field $\boxed{a(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in \mathcal{A}[[z,z^{-1}]] \subset \mathcal{U}(\mathcal{A})[[z,z^{-1}]].}$

Key property: $\forall v\in F_\mu \;\exists N: a_{\geq N} v = 0 \;\Rightarrow\; \boxed{a(z): F_\mu \longrightarrow F_\mu((z))}$

Note that the defining relation of $\mathcal{A}$ implies the following formula:

$$\boxed{[a(z), a(w)] = \sum_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1} = \underbrace{\partial_w \delta(w-z) = -\partial_z \delta(w-z)}_{=:\delta'(w-z)}}$$

which holds both as an equality in $\mathcal{U}(\mathcal{A})_{(C-1)}[[z,z^{-1},w,w^{-1}]]$ and in $\mathrm{Hom}(F_\mu, F_\mu((z))((w)))$.

Likewise, we also have

$$a(z)a(w) - :a(z)a(w): = \sum_{n\geq 0} [a_n, a_{-n}] z^{-n-1} w^{n-1} \underset{C=1}{=\!=} \sum_{n>0} n z^{-n-1} w^{n-1} = \frac{1}{z^2}\cdot \underbrace{\left(1-\frac{w}{z}\right)^{-2}}_{\substack{\text{expanded} \\ \text{in non-negative powers of } \frac{w}{z}}} = \left(\frac{1}{z-w}\right)^2.$$

So: $\boxed{a(z)a(w) = :a(z)a(w): + \frac{1}{(z-w)^2}}$

can think of as a "regular part". can think of as a "singular part"

- Let us now encode all the generators $\{L_n\}_{n\in\mathbb{Z}}$ of the Witt algebra W into

$$\boxed{T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}}$$

The defining relations in W imply

$$[T(z), T(w)] = \sum_{n,m\in\mathbb{Z}} (n-m) \underbrace{L_{n+m}}_{=:k} z^{-n-2} w^{-n-2} = \sum_{k\in\mathbb{Z}} L_k \sum_{m\in\mathbb{Z}} (k-2m) z^{m-k-2} w^{-m-2}$$

$$= \Big(\sum_{k\in\mathbb{Z}} L_k\cdot (k+2) z^{-k-3}\Big)\Big(\sum_{m\in\mathbb{Z}} z^{m+1} w^{-m-2}\Big) - 2\Big(\sum_{k\in\mathbb{Z}} L_k z^{-k-2}\Big)\Big(\sum_{m\in\mathbb{Z}} (m+1) z^m w^{-m-2}\Big)$$

So: $\boxed{[T(z), T(w)] = -T'(z)\cdot \delta(w-z) + 2T(z)\cdot \delta'(w-z)}$ Exercise: Verify that this relation implies the rels in W

• Finally, we also encode all $\{L_n\}_{n\in\mathbb{Z}}$ of Virasoro alg. Vir into $\boxed{T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}}$

Then, the formula for $[T(z), T(w)]$ is obtained from that for W by adding
$$\sum_{n\in\mathbb{Z}} \frac{n^3-n}{12} C z^{-n-2} w^{-n-2} = \frac{C}{12}\partial_w^3 \delta(w-z) = \frac{C}{12}\delta'''(w-z)$$

So: $\boxed{[T(z), T(w)] = -T'(z)\cdot\delta(w-z) + 2T(z)\cdot\delta'(w-z) + \frac{C}{12}\delta'''(w-z)}$

Exercise: Verify the converse, i.e. that this $\underset{1+C\text{-central}}{\text{single rel-n}}$ implies all rels in Vir

Recall now the construction of $\mathrm{Vir}\ltimes\mathcal{A} \curvearrowright F_\mu$ from last time. We shall view now L_n, a_m as endomorphisms of F_μ. Then:

- the relation $[L_n, a_m] = -m a_{n+m}$ $\forall n,m$ (see Lemma 1 of Lecture 5) can be encoded by a single equality in $\mathrm{Hom}(F_\mu, F_\mu((z))((w)))$:
$$[T(z), a(w)] = \sum_{n,m} -m \underbrace{a_{n+m}}_{k} z^{-n-2} w^{-m-1} = \sum_{k\in\mathbb{Z}} a_k \sum_{m\in\mathbb{Z}} (-m) z^{m-k-2} w^{-m-1}$$
$$= \Big(\sum_{k\in\mathbb{Z}} a_k z^{-k-1}\Big)\Big(\sum_{m\in\mathbb{Z}} (-m) z^{m-1} w^{-m-1}\Big) = a(z)\delta'(w-z)$$
$$\Downarrow$$
$$\boxed{[T(z), a(w)] = a(z)\delta'(w-z)}$$

- the defining formula for the corresponding $\mathrm{Vir}\curvearrowright F_\mu$ (note: $\lambda=0$ in that discussion) is equivalent to the following simple formula:
$$\boxed{T(z) = \frac{1}{2} :a(z)^2:}$$

Rmk 2: As before, we note that $:a(z)^2:$ is a well-defined element of $\mathrm{Hom}(F_\mu, F_\mu[[z,z^{-1}]])$ [unlike the unordered square $a(z)^2$, which does not make sense]

Exercise: (a) Verify that $\forall \beta\in\mathbb{C}$, the formula
$$T(z) = \frac{1}{2}:a(z)^2: + \beta\cdot a'(z)$$
defines a representation of Vir on F_μ with $c = 1-12\beta^2$.

(b) Verify that
$$L_n \mapsto L_n + \beta a_n \ (n\neq 0),\quad L_0 \mapsto L_0 + \beta a_0 + \frac{\beta^2}{2}K,\quad C\mapsto C$$
defines a homomorphism $\varphi_\beta : \mathrm{Vir} \to \mathrm{Vir}\ltimes\mathcal{A}$.

(c) Show that twisting repr-n of part (a) by the homom. of part (b), we obtain the representation $\mathrm{Vir}\curvearrowright F_\mu$ with $\lambda = \beta/i$ from [Lecture 5, Prop. 1].

Our next discussion will concern repr-s of gl_∞ and $\mathfrak{o}_\infty$.

Def 2: gl_∞ is the Lie algebra of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ s.t. all but a finite number of a_{ij} are 0, with the Lie bracket being the usual commutator.

- Note that gl_∞ has a basis $\{E_{ij}\}_{i,j\in\mathbb{Z}}$ (E_{ij} has 1 at the $(i,j)^{th}$ entry and 0's elsewhere). with the commutator defined by $[E_{ij}, E_{mn}] = \delta_{jm} E_{in} - \delta_{in} E_{mj}$
- This algebra acts naturally on the vector space $V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C} v_j$ via usual f-la:

$$\boxed{E_{ij}(v_k) = \delta_{jk}\cdot v_i} \quad \leftarrow \text{multiplying an } \underset{\left(\text{with almost all entries}=0\right)}{\infty\text{-matrix}} \text{ by } \underset{\left(\text{with almost all entries}=0\right)}{\infty\text{-column}}$$

Consequently, gl_∞ also acts on $S^m V$, $\Lambda^m V$.

- Note that gl_∞ may be viewed as a $\mathbb{Z}$-graded Lie alg. with $\deg(E_{ij}) = j-i$, so that

$$\boxed{gl_\infty = \underset{\substack{\text{strictly}\\ \text{lower-}\\ \text{triangular}}}{\mathfrak{n}_-} \oplus \underset{\text{diagonal}}{\mathfrak{h}} \oplus \underset{\text{strictly upper-triangular matrices}}{\mathfrak{n}_+}}$$

- $\mathfrak{h}$-abelian $\Rightarrow$ any $\lambda\in\mathfrak{h}^*$ defines a Verma module M_λ and its irreducible quotient L_λ where $L_\lambda = M_\lambda / J_\lambda$ with $J_\lambda = \mathrm{Ker}(\ ,\ )_\lambda$ as always.
- There is an antilinear antiinvolution $\dagger : gl_\infty \to gl_\infty$ given by $E_{ij}^\dagger = E_{ji}$
- V is a unitary gl_∞-module, hence, so are all $S^m V$, $\Lambda^m V$.

__However__: In contrast to the previous discussions, none of $V, S^m V, \Lambda^m V$ have highest weights.

__Exercise__: Prove that all repr-s $\{S^m V, \Lambda^m V\}_{m\geq 1}$ are irreducible gl_∞-representations.

__Rmk 3__: This can be generalized to any __Schur modules of V__. The latter are defined as follows. Pick $n\geq 1$ and $\pi\in \mathrm{Irr}(S_n)$ - irred. repr-n of Symmetric group. We define a representation $\underset{\uparrow\ \pi\text{-th Schur module of } V.}{S_\pi(V)}$ of gl_∞ via $\boxed{S_\pi(V) := \mathrm{Hom}_{S_n}(\pi, V^{\otimes n})}$

Then, the above exercise can be generalized as follows:

__Exercise__: For all n, π, $S_\pi(V)$ is an irreducible gl_∞-representation.

__Hint__: Reduce to the known Schur-Weyl duality in fin.dim. setting.

__Next Goal__: To "merry" two different worlds:
- Schur modules of V, in particular, $\Lambda^m V$
- category $\mathcal{O}^+$

This shall be achieved via __semi-infinite wedges__.

Def 3: (a) An *elementary $\frac{\infty}{2}$-wedge* is a formal infinite wedge product $v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots$ with integers $i_0>i_1>i_2>\ldots$ satisfying $i_{k+1}=i_k-1$ for $k\gg 1$.

(b) The *semiinfinite wedge space* $\Lambda^{\frac{\infty}{2}}V$ is the $\mathbb{C}$-vector space with the basis given by elementary semiinfinite wedges

Note: $\boxed{\Lambda^{\frac{\infty}{2}}V=\bigoplus_{m\in\mathbb{Z}}\Lambda^{\frac{\infty}{2},m}V, \text{ where } \Lambda^{\frac{\infty}{2},m}V=\mathrm{span}\{v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots \mid i_k=m-k \text{ for } k\gg 0\}}$

Rmk 4: Let me point out that this definition of $\Lambda^{\frac{\infty}{2},m}V$ depends on the choice of a basis of V.

Proposition 2: The usual Leibniz rule defines an action $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V$

$$a(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots)=\sum_{k=0}^{\infty}v_{i_1}\wedge\ldots\wedge v_{i_{k-1}}\wedge a(v_{i_k})\wedge v_{i_{k+1}}\wedge\ldots$$

Down-to-earth, this action is given via reorder gaining $\pm$sign.

$$\boxed{\underset{i_0>i_1>\ldots\ \&\ i_k=m-k \text{ for } k\gg 0}{E_{rs}(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots)}=\begin{cases}v_{i_0}\wedge\ldots\wedge v_{i_{k-1}}\wedge v_r\wedge v_{i_{k+1}}\wedge\ldots & \text{if } i_k=s\\ 0 & \text{if } s\notin\{i_0,i_1,i_2,\ldots\}\end{cases}}$$

Exercise: Prove Prop 2, i.e. verify that these formulas define an action $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V$.

Note: The conditions $i_0>i_1>i_2>\ldots$ & $i_k=m-k$ for $k\gg 0$ imply that the sequence $\{i_k+k-m\}_{k=0}^{\infty}$ is a partition, i.e. it is non-increasing and all but finitely many terms are 0.

Def 4: Define a grading on $\Lambda^{\frac{\infty}{2},m}V$ via

$$\Lambda^{\frac{\infty}{2},m}V=\bigoplus_{d\in\mathbb{Z}_{\le 0}}(\Lambda^{\frac{\infty}{2},m}V)[d], \text{ where } (\Lambda^{\frac{\infty}{2},m}V)[d]=\mathrm{span}\left\{v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots \,\middle|\, \begin{array}{l} i_0>i_1>\ldots\\ i_k=m-k \text{ for } k\gg 0\\ \sum_{k=0}^{\infty}(i_k+k-m)=-d\end{array}\right\}$$

i.e. $\boxed{\deg(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots):=-\sum_{k=0}^{\infty}(i_k+k-m)}$

The observation made before the definition implies that the basis of $(\Lambda^{\frac{\infty}{2},m}V)[d]$ is parametrized by all partitions of $-d$ ($\forall d\in\mathbb{Z}_{\le 0}$).

Lemma 2: (a) $\Lambda^{\frac{\infty}{2},m}V$ is a graded $\mathfrak{gl}_\infty$-module w.r.t. the grading on $\mathfrak{gl}_\infty$ defined before.

(b) Consider $\boxed{\psi_m:=v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots\in(\Lambda^{\frac{\infty}{2},m}V)[0]}$.

Then: $\mathfrak{n}_+(\psi_m)=0$ and $h(\psi_m)=\omega_m(h)\cdot\psi_m$ $\forall h\in\mathfrak{h}$, where

$\omega_m=(\ldots,1,1,1,0,0,0,\ldots)$ - weight of $\mathfrak{gl}_\infty$.
$\quad\ldots\ m-2\ \ m-1\ \ m\ \ m+1\ \ m+2\ldots$

(c) ψ_m generates $\Lambda^{\frac{\infty}{2},m}V$.

► Obvious ∎

Proposition 3: For any $m\in\mathbb{Z}$, $\Lambda^{\frac{\infty}{2},m}V$ is an irreducible highest weight repr-n L_{ω_m} of $\mathfrak{gl}_\infty$. Moreover, L_{ω_m} is unitary.

► As we discussed last time, it suffices to check that $\Lambda^{\frac{\infty}{2},m}V$ is unitary (as it is h.wt by Lemma 2). The latter is obvious: the Hermitian form in which $\frac{\infty}{2}$ wedges of $\Lambda^{\frac{\infty}{2},m}V$ form an orthonormal basis is unitary and $\dagger$-invariant, i.e. $(E_{ij}w_1,w_2)=(w_1,E_{ji}w_2)$ ∎

Corollary 2: If $\lambda=(\lambda_i)_{i\in\mathbb{Z}}$, $\lambda_i\in\mathbb{R}$, $\lambda_{i+1}-\lambda_i\in\mathbb{Z}_{\ge 0}$ and is ZERO if $|i|\gg 1$, then L_λ is unitary.

► If $\beta_i=\gamma\in\mathbb{R}$ $\forall i$, then L_β is 1-dim ($X\mapsto\gamma\cdot\mathrm{tr}(X)$) and is clearly unitary.

Any λ as above can be written as $\beta+\sum_j n_j\omega_j$, $n_j\in\mathbb{Z}_{\ge 0}$ almost all ZERO $\Rightarrow$ L_λ - summand in $L_\beta\otimes\bigotimes_j L_{\omega_j}^{\otimes n_j}$ $\Rightarrow$ unitary ∎