— LECTURE 7 —

- Last time: $\mathfrak{gl}_\infty$

$\mathfrak{gl}_\infty \curvearrowright S^mV,\ \Lambda^mV,\ S_\pi(V)$ — no highest weight vectors

$\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V \ni \psi_m := v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$ — highest weight vector of h.wt $\omega_m = (\ldots, \underset{m-1}{1}, \underset{m}{1}, \underset{m+1}{0}, 0, \ldots 0, \ldots)$

Prop1: For any $m \in \mathbb{Z}$, $\Lambda^{\frac{\infty}{2},m}V$ is an irreducible h.wt. repr-n L_{ω_m} of $\mathfrak{gl}_\infty$.
Moreover, L_{ω_m} is unitary.

► It is clear that $\Lambda^{\frac{\infty}{2},m}V$ is generated by its highest weight vector ψ_m. Hence, it suffices to show $\Lambda^{\frac{\infty}{2},m}V$ is unitary (since as we know a unitary high.wt. repr-n must be irreduc). But the latter is obvious: the Hermitian form in which $\frac{\infty}{2}$ wedges of $\Lambda^{\frac{\infty}{2},m}V$ form an orthonormal basis is unitary and †-invariant, i.e. $(E_{ij}w_1, w_2) = (w_1, E_{ji}w_2)$ ■

Corollary1: If $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$, $\lambda_i \in \mathbb{R}$, $\lambda_{i+1} - \lambda_i \in \mathbb{Z}_{\geq 0}$ and is ZERO if $|i| \gg 0$, then L_λ is unitary

► If $\beta = (\beta_i)_{i\in\mathbb{Z}}$ with all $\beta_i \overset{\forall i}{=} \nu \in \mathbb{R}$, then L_β is a 1-dim $\mathfrak{gl}_\infty$-repr. $(X \mapsto \nu \cdot \mathrm{Tr}(X))$ which is clearly unitary (note: ν must be real!) Any λ as above may be written as $\beta + \sum_i n_i \omega_i$ where $n_i \in \mathbb{Z}_{\geq 0}$ and almost all of them are ZERO.
But then L_λ is a submodule of the unitary repr-n $L_\beta \otimes \bigotimes_i L_{\omega_i}^{\otimes n_i}$ $\Rightarrow$ unitary itself ■

Prop2: If an irreducible repr-n L_λ of $\mathfrak{gl}_\infty$ is unitary, and $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$ with $\lambda_i = \lambda_+ \in \mathbb{R}$ for $i \gg 0$, $\lambda_i = \lambda_- \in \mathbb{R}$ for $i \ll 0$ (i.e. λ "stabilizes at $\pm\infty$"), then λ must be of the form as in Cor 1, i.e. $\lambda_{i+1} - \lambda_i \in \mathbb{Z}_{\geq 0}$ $\forall i$.

► Consider the $\mathfrak{sl}_2$-submodule generated by v_λ over $\mathfrak{sl}_2^{(i)} = \langle E_{i,i+1}, E_{i+1,i}\rangle$. Then, it is a highest weight unitary $\mathfrak{sl}_2$-representation of highest weight $\mu_i = \lambda_i - \lambda_{i+1}$ (hence $\simeq L_{\mu_i}^{(\mathfrak{sl}_2)}$)
Now the condition $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ follows from the following simple result:

Lemma1: A unitary representation L_μ of $\mathfrak{sl}_2$ must have $\mu \in \mathbb{Z}_{\geq 0}$.

► As we computed the $\mathfrak{sl}_2$-invariant form on L_μ: on level n it gave
$(f^n v_\mu, f^n v_\mu) = \underline{n!\,\mu(\mu-1)\cdot \ldots \cdot(\mu-n+1)}$. If $\mu \notin \mathbb{Z}_{\geq 0}$, then $L_\mu \simeq M_\mu \Rightarrow n$ - any nonnegative integer $\Rightarrow$ not all these numbers are positive real! Instead, if $\mu \in \mathbb{Z}_{\geq 0}$, then $0 \leq n \leq \mu$ and all these numbers $\in \mathbb{R}_{>0}$

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• Today: $\overline{a}_\infty$ and a_∞.

Def 1: $\overline{a}_\infty$ is the Lie algebra of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many nonzero diagonals (i.e. $\exists N\ \forall i,j$ s.t. $|i-j|>N$ $a_{ij}=0$), with the Lie bracket being the usual commutator

Remark 1: (i) $gl_\infty \subset \overline{a}_\infty$

(ii) $\mathbb{1}\notin gl_\infty$, but $\mathbb{1}\in\overline{a}_\infty$

(iii) $\overline{a}_\infty$ is still $\mathbb{Z}$-graded $\boxed{\overline{a}_\infty=\bigoplus_{i\in\mathbb{Z}}\overline{a}_\infty^i}$

the subspace of matrices $(a_{\alpha\beta})$ s.t. $a_{\alpha\beta}=0$ unless $\beta-\alpha=i$.

(iv) this gives a triangular decomposition $\overline{a}_\infty=n_-\oplus\mathfrak{h}\oplus n_+$, $\mathfrak{h}$-abelian.

(v) For $A,B\in\overline{a}_\infty$, the product AB is well-defined and belongs to $\overline{a}_\infty$

For $A\in\overline{a}_\infty$, $B\in gl_\infty$, AB is well-defined and belongs to gl_∞.

(vi) $\{E_{ij}\}_{i,j\in\mathbb{Z}}$ is no longer a vector space basis!

(vii) $\overline{a}_\infty$ is not of countable dimension!

Def 2: Let $T: V\to V$ be the shift operator, defined via $T(v_i)=v_{i-1}\ \forall i$

The shift operator T as well as any integer power of it may be viewed as an element of $\overline{a}_\infty$: $\boxed{T^k=\sum_{i\in\mathbb{Z}}E_{i,i+k}\in\overline{a}_\infty^k\quad\forall k\in\mathbb{Z}}$

Remark 2: From this perspective, we may think about $\overline{a}_\infty$ as of the algebra of difference operators, i.e. formal expressions $\underbrace{A=\sum_{k=\alpha}^{\beta}\gamma_k(n)T^k}_{(Ax)_n=\sum_{k=\alpha}^{\beta}\gamma_k(n)x_{n+k}\ \forall x\in V,\, n\in\mathbb{Z}}$, where $\alpha\le\beta\in\mathbb{Z}$ and $\gamma_k:\mathbb{Z}\to\mathbb{C}$

Question: Can we extend $\boxed{gl_\infty \overset{\rho}{\curvearrowright} \Lambda^{\frac{\infty}{2},m}V}$ to an action $\overline{a}_\infty\curvearrowright\Lambda^{\frac{\infty}{2},m}V$?

Let us try to define such an action in the most naive way: just as for gl_∞.

• If $i\neq 0$, then any element $A\in\overline{a}_\infty^i$ has the form $A=\sum_{j\in\mathbb{Z}}a_jE_{j,j+i}$ and it is easy to see that its action on any elementary $\frac{\infty}{2}$-wedge $v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots$ is well-defined (indeed: - if j is big enough so that $j+i>i_0$ $\Rightarrow$ all summands disappear

- if j is small enough (explain how) then applying A to v_{i_ℓ} we shall get wedge with some v_k appearing twice $\Rightarrow$ ZERO)

• However, if $i=0$ and $A=\sum_{j\in\mathbb{Z}}a_jE_{jj}\in\overline{a}_\infty^0$, then there is a problem!

(indeed: $A(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots)=(\underbrace{a_{i_0}+a_{i_1}+a_{i_2}+\ldots}_{\text{does not have to contain only finitely many nonzero el-s.}})\cdot v_{i_0}\wedge v_{i_1}\wedge\ldots$)

To fix: we shall redefine the action of $\{E_{ii}\}_{i\le 0}$

Define $$\boxed{\hat{\rho}(E_{ij})=\begin{cases}\rho(E_{ij}) & \text{unless } i=j\le 0\\ \rho(E_{ij})-1 & \text{if } i=j\le 0\end{cases}}$$

This gives rise to a linear map

$$\boxed{\hat{\rho}: \overline{\mathfrak{a}}_\infty \longrightarrow \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V) \text{ via } (a_{ij})_{i,j\in\mathbb{Z}} \longmapsto \sum a_{ij}\hat{\rho}(E_{ij})}$$

Exercise: Explain why $\hat{\rho}(A)$ is well-defined $\forall A\in\overline{\mathfrak{a}}_\infty$.

! BUT: $\hat{\rho}$ is NOT a Lie algebra homomorphism.

In other words, $[\hat{\rho}(A),\hat{\rho}(B)]-\hat{\rho}([A,B])\neq 0$. But in what follows, we shall actually see that it is scalar operator.

Let us write A, B as 2×2 block matrices $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, $B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}$, where the division is w.r.t. $i\leq 0$ and $i>0$.

Prop 3: Let $\alpha(A,B):=-\hat{\rho}([A,B])+[\hat{\rho}(A),\hat{\rho}(B)]$ $\forall A,B\in\overline{\mathfrak{a}}_\infty$.

Then: $\boxed{\alpha(A,B)=\mathrm{Tr}(-B_{12}A_{21}+A_{12}B_{21})=\mathrm{Tr}(-A_{21}B_{12}+B_{21}A_{12})}$ (#)

Remark 3: The right-hand side of (#) is well-defined as each of $A_{12}, A_{21}, B_{12}, B_{21}$ has only finitely many nonzero elements.

Exercise (Hwk 4): Prove Proposition 3.

As an immediate corollary of Proposition 3, we obtain:

Lemma 2: The bilinear map $\boxed{\alpha: \overline{\mathfrak{a}}_\infty\times\overline{\mathfrak{a}}_\infty\longrightarrow\mathbb{C}}$ defined by (#) is a 2-cocycle on $\overline{\mathfrak{a}}_\infty$, which is nontrivial (i.e. not a 2-coboundary).

Japanese cocycle (after Date–Jimbo–Kashiwara–Miwa)

- The cocycle condition follows immediately from the fact that $\alpha(A,B)=[\hat{\rho}(A),\hat{\rho}(B)]-\hat{\rho}([A,B])$

Exercise: Explain why!

- To see that α is nontrivial, consider elements $\{T^k\}_{k\in\mathbb{Z}}\subset\overline{\mathfrak{a}}_\infty$.

Then $\alpha(T^n,T^m)=n\delta_{n,-m}$, hence, restricting to the abelian Lie subalg. $\overline{\mathcal{A}}$ spanned by $\{T^k\}_{k\in\mathbb{Z}}$, we obtain the Heisenberg extension, which is nontrivial (by Lecture 1)

Remark 4: Note that $\alpha|_{\mathfrak{gl}_\infty}$ is a 2-coboundary, since for $A,B\in\mathfrak{gl}_\infty$, we have

$$\alpha(A,B)=\mathrm{Tr}(J\cdot\underbrace{[A,B]}_{\begin{pmatrix}[A_{11},B_{11}]+A_{12}B_{21}-B_{12}A_{21} & * \\ * & *\end{pmatrix}}), \text{ where } J=\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}$$

Remark 5: Explicitly, we get $\boxed{\alpha((a_{ij}),(b_{ij}))=\sum_{i\leq 0<j}a_{ij}b_{ji}-\sum_{j\leq 0<i}a_{ij}b_{ji}=\sum_{i,j}a_{ij}b_{ji}(\delta_{j>0}-\delta_{i>0})}$

Def 3: Let $\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}\cdot K$ be the 1-dim central extension of $\bar{\mathfrak{a}}_\infty$ given by 2-cocycle α.

Summarizing Prop 3 & Lemma 2, we immediately get the final key result:

Theorem 1: Let $\hat{\varrho}: \mathfrak{a}_\infty \to \mathrm{End}(\Lambda^{\frac{\infty}{2},m} V)$ be the linear operator s.t. $\hat{\varrho}(K) = \mathrm{Id}$, while $\hat{\varrho}|_{\bar{\mathfrak{a}}_\infty}$ coincides with the previous construction. Then, it is a Lie alg. homom.

In other words, $\boxed{\mathfrak{a}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V}$

Remark 6: (i) $\mathfrak{a}_\infty$ is $\mathbb{Z}$-graded ($\deg(K)=0$, while $\mathbb{Z}$-grading on $\bar{\mathfrak{a}}_\infty$ was defined before)

(ii) $\Lambda^{\frac{\infty}{2},m} V$ is a $\mathbb{Z}$-graded module over $\mathfrak{a}_\infty$.

(iii) The Lie alg. embedding $\mathcal{A} \hookrightarrow \bar{\mathfrak{a}}_\infty$ gives rise to a Lie alg. embedding

$$\boxed{\underbrace{\mathcal{A}}_{\text{Heisenberg/oscillator alg}} \hookrightarrow \mathfrak{a}_\infty \quad \text{via} \quad a_n \mapsto T^n,\ K \mapsto K}$$

(iv) Thus, we get the action of Heisenberg alg. $\mathcal{A} \curvearrowright \Lambda^{\frac{\infty}{2},m} V$.

Let us define weight $\bar{\omega}_m \in (\mathfrak{a}_\infty[0])^*$ as follows:

$$\boxed{\bar{\omega}_m : \quad K \mapsto 1, \quad \sum_{i\in\mathbb{Z}} a_i E_{ii} \mapsto \begin{cases} \sum_{j=1}^{m} a_j, & \text{if } m \geq 0 \\ -\sum_{j=m+1}^{0} a_j, & \text{if } m<0 \end{cases}}$$

Prop 4: $\Lambda^{\frac{\infty}{2},m} V$ is the irreducible highest weight repr. $L_{\bar{\omega}_m}$ of $\mathfrak{a}_\infty$. Moreover, $L_{\bar{\omega}_m}$ is unitary.

▸ Analogous to Prop 1 ∎

Corollary 2: For any collection $(n_i)_{i\in\mathbb{Z}}$ of integers, almost all of which are ZERO, the $\mathfrak{a}_\infty$-representation $L_{\sum n_i \bar{\omega}_i}$ is unitary!

▸ Analogous to Corollary 1 ∎

As noted in Rmk 6 (iii), we have a natural embedding $\mathcal{A} \hookrightarrow \mathfrak{a}_\infty$. It turns out that $\mathfrak{a}_\infty$ also contains Virasoro algebras inside itself (and in many different ways). To specify the latter explicitly, recall the action $W \curvearrowright V_{\alpha,\beta} = \{g(t)t^\alpha (dt)^\beta \mid g \in \mathbb{C}[t,t^{-1}]\}$ from [Homework 1, Problem 3]. Switching $k \to -k$ in part (b) of that problem, we see that $V_{\alpha,\beta}$ has a basis $\{v_k\}_{k\in\mathbb{Z}}$ with the action of the Witt alg. given by

$$\boxed{L_n(v_k) = (k - \alpha - (n+1)\beta)\cdot v_{k-n} \quad \forall k \in \mathbb{Z},\ n \in \mathbb{Z}}$$

Thus, we get a Lie algebra embedding

$$\boxed{\bar{\varphi}_{\alpha,\beta} : \underbrace{W}_{\text{witt alg.}} \hookrightarrow \bar{\mathfrak{a}}_\infty \quad \text{via} \quad L_n \mapsto \sum_{k\in\mathbb{Z}} (k-\alpha-(n+1)\beta) E_{k-n,n}}$$

Exercise (Hwk 4): Viewing L_n as elements of $\mathfrak{a}_\infty$, we have $[L_n, L_m] = (n-m)L_{n+m} + \alpha(L_n, L_m)$.

Prove that $\boxed{\alpha(L_n, L_m) = \delta_{n,-m}\left(\frac{n^3-n}{12}c_\beta + 2n\cdot h_{\alpha,\beta}\right), \text{ where } c_\beta = -12\beta^2+12\beta-2,\ h_{\alpha,\beta} = \frac{\alpha(\alpha+2\beta-1)}{2}}$

Noticing that $(L_n, L_m) \mapsto 2n\delta_{n,-m}\cdot h_{\alpha,\beta}$ is a 2-coboundary, while the first summand is a c_β-multiple of the Virasoro 2-cocycle, we obtain:

Prop 5: For any $\alpha,\beta \in \mathbb{C}$, there is a Lie algebra embedding

$$\boxed{\varphi_{\alpha,\beta}: Vir \hookrightarrow \mathfrak{a}_\infty \quad \text{s.t.} \quad L_n \mapsto \overline{\varphi}_{\alpha,\beta}(L_n) + \delta_{n,0}\cdot h_{\alpha,\beta}\cdot K,\ C \mapsto c_\beta\cdot K}$$

Restricting $\mathfrak{a}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V$ to Vir (embedded via $\varphi_{\alpha,\beta}$), we get $\underbrace{Vir \curvearrowright \Lambda^{\frac{\infty}{2},m}V}_{\text{Virasoro module of } \frac{\infty}{2} \text{ forms}}$

Exercise (Hwk 4): Verify that $\boxed{\begin{array}{l} L_k(\psi_m) = 0 \text{ for } k>0 \\ L_0(\psi_m) = \frac{(\alpha-m)(\alpha+2\beta-1-m)}{2}\psi_m. \end{array}}$

Corollary 3: For $\alpha,\beta \in \mathbb{C}$, set $\lambda := \left(\frac{(\alpha-m)(\alpha+2\beta-1-m)}{2}, -12\beta^2+12\beta-2\right)$. Then we get a Virasoro module homomorphism

$$\boxed{\begin{array}{ccc} M_\lambda & \longrightarrow & \Lambda^{\frac{\infty}{2},m}V \\ \Psi & & \Psi \\ v_\lambda & \longmapsto & \psi_m \end{array}}$$

$\leftarrow$ better to write $\varphi^*_{\alpha,\beta}(\Lambda^{\frac{\infty}{2},m}V)$ or simply by $\Lambda^{\frac{\infty}{2},m}V_{\alpha,\beta}$.

Let us now get back to the action $\underbrace{\mathcal{A} \curvearrowright \Lambda^{\frac{\infty}{2},m}V}_{\text{this makes } \Lambda^{\frac{\infty}{2},m}V \text{ into the } \mathbb{Z}\text{-graded } \mathcal{A}\text{-module}}$.

Recall that $\deg(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots) = -\sum_{k\geq 0}(i_k+k-m) =: -d$, where last time we noticed that $\{i_k+k-m\}_{k\geq 0}$ is a non-increasing sequence of non-negative integers stabilizing to ZERO. The latter defines a bijection b/w elementary $\frac{\infty}{2}$-wedges in $\Lambda^{\frac{\infty}{2},m}V$ and partitions, which also identifies the degree of the $\frac{\infty}{2}$-wedge with $-$(sum of el-s of the partition).

Hence: $$\boxed{ch_{\Lambda^{\frac{\infty}{2},m}V}(q) := \sum \dim\left(\Lambda^{\frac{\infty}{2},m}V[-d]\right)q^d = \sum_{d\geq 0} p(d)q^d = \frac{1}{(1-q)(1-q^2)(1-q^3)\ldots}}$$

Prop 6: For any $m\in\mathbb{Z}$, $\Lambda^{\frac{\infty}{2},m}V \simeq F_m$ as $\mathcal{A}$-representations.

Recall that $\begin{array}{l} \overset{\mathcal{A}}{\underset{}{a_n}} \mapsto T^n \in \mathfrak{a}_\infty \\ K \mapsto K \end{array}$. It is clear that $K(\psi_m) = 1\cdot\psi_m$ and $T^n(\psi_m) = 0\ \forall n>0$.

Moreover, $T^0(\psi_m) = (\sum_i E_{ii})(\psi_m) = m\cdot\psi_m$ (treat separately the cases $m\geq 0$ and $m<0$).

Hence, we have an $\mathcal{A}$-module homom. $\sigma_m: \begin{array}{ccc} F_m & \longrightarrow & \Lambda^{\frac{\infty}{2},m}V \\ 1 & \longmapsto & \psi_m \end{array}$ which also preserves $\mathbb{Z}$-grading.

Since F_m is irreducible and $ch_{F_m}(q) = ch_{\Lambda^{\frac{\infty}{2},m}V}(q)$, we immediately get σ_m- isom. $\blacksquare$