# — LECTURE 8 —

Last time we concluded with the isomorphism of $\mathcal{A}$-modules $\sigma_m : \Lambda^{\frac{\infty}{2},m} V \xrightarrow{\sim} F_m \ \forall m \in \mathbb{Z}$.

Define:

$$\boxed{\begin{array}{ll} \mathcal{B}^{(m)} := F_m = \{\text{the space of polynomials in } x_1, x_2, \ldots\} & \leftarrow \underline{\text{bosonic space}} \\ \mathcal{F}^{(m)} := \Lambda^{\frac{\infty}{2},m} V = \{\text{the space spanned by elementary } \tfrac{\infty}{2}\text{-wedges}\} & \leftarrow \underline{\text{fermionic space}} \\ \mathcal{F} := \bigoplus_{m \in \mathbb{Z}} \mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2}} V & \\ \mathcal{B} := \bigoplus_{m \in \mathbb{Z}} \mathcal{B}^{(m)} = \mathbb{C}[z, z^{-1}; x_1, x_2, x_3, \ldots] & (\text{here identify } \mathcal{B}^{(m)} \text{ with } z^m \mathbb{C}[x_1, x_2, \ldots]) \end{array}}$$

Then, we obtain an $\mathcal{A}$-module isomorphism

$$\boxed{\sigma = \bigoplus_m \sigma_m : \mathcal{F} \xrightarrow{\sim} \mathcal{B}} \quad \leftarrow \underline{\text{Boson-Fermion Correspondence}}$$

There are two natural questions to ask:

<u>Q1</u>: Which polynomials arise as the images of elementary $\frac{\infty}{2}$-wedges?

<u>Q2</u>: How to extend $\mathcal{A} \curvearrowright \mathcal{B}$ to $\mathfrak{a}_\infty \curvearrowright \mathcal{B}$?

First, we shall treat the second question!

**Def 1**: (a) For $i \in \mathbb{Z}$, define the <u>wedging operator</u> $\xi_i = \hat{v}_i : \mathcal{F} \to \mathcal{F}$ by $\hat{v}_i(\psi) = v_i \wedge \psi$

(so that $\hat{v}_i : \mathcal{F}^{(m)} \to \mathcal{F}^{(m+1)}$)

(b) For $i \in \mathbb{Z}$, define the <u>contraction operators</u> $\xi_i^* = \check{v}_i : \mathcal{F} \to \mathcal{F}$ via the contraction with $v_i$, i.e.

$$\check{v}_i(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) = \begin{cases} 0, & \text{if } i \notin \{i_0, i_1, i_2, \ldots\} \\ (-1)^k \cdot v_{i_0} \wedge \ldots \wedge v_{i_{k-1}} \wedge v_{i_{k+1}} \wedge \ldots & \text{if } i = i_k \end{cases} \quad \text{if } i_0 > i_1 > i_2 > \ldots$$

(so that $\check{v}_i : \mathcal{F}^{(m)} \to \mathcal{F}^{(m-1)}$)

<u>Exercise</u>: Understand this definition.

<u>Lemma 1</u>: For any $i, j \in \mathbb{Z}$, we have:

$$\boxed{\hat{v}_i \hat{v}_j + \hat{v}_j \hat{v}_i = 0, \quad \check{v}_i \check{v}_j + \check{v}_j \check{v}_i = 0, \quad \check{v}_i \hat{v}_j + \hat{v}_j \check{v}_i = \delta_{ij}}$$

<u>Exercise</u>: Prove this Lemma.

From the very definition of $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2}} V$, we get

$$\boxed{\rho(E_{ij}) = \xi_i \xi_j^*}$$

$$\Downarrow$$

$\hat{\rho}(E_{ij})$ introduced last time equals $\boxed{\hat{\rho}(E_{ij}) = \begin{cases} \xi_i \xi_j^* - 1, & \text{if } i = j \leq 0 \\ \xi_i \xi_j^*, & \text{elsewhere} \end{cases} \overset{\text{Def}}{=\!=} :\xi_i \xi_j^*:}$

Recalling that under the embedding $\mathcal{A} \hookrightarrow \mathfrak{a}_\infty$, we have $a_k \mapsto \sum_i E_{i,i+k}$

$\Rightarrow \boxed{\hat{\rho}(a_k) = \sum_i :\xi_i \xi_{i+k}^*:} \leftarrow$ the way $k^{th}$ generator $a_k$ of the Heisenberg alg. acts on $\Lambda^{\frac{\infty}{2}} V$.

We may encode this (over all $k \in \mathbb{Z}$) by $\boxed{\hat{\rho}(a(z)) = :\xi(z) \xi^*(z):}$ where $\xi(z) = \sum \xi_n z^{-n-1/2}$

$\xi^*(z) = \sum \xi_n^* z^{-n-1/2}$

To answer Q2, it suffices to express the action of $E_{ij} \in \mathfrak{a}_\infty$, given by $\hat{\varrho}(E_{ij})$, via the action of $a_k$, given by $\hat{\varrho}(a_k)$. But by the previous formula, it suffices to express $\xi_i, \xi_j^*$ via $\hat{\varrho}(a_k)$. The latter shall be achieved by the vertex operator construction.

<u>Def 2</u>: Define the following quantum fields

$$\boxed{\begin{aligned} X(u) &:= \sum_{n\in\mathbb{Z}} \xi_n u^n \in \mathrm{End}(\mathcal{F})[[u,u^{-1}]] \\ X^*(u) &:= \sum_{n\in\mathbb{Z}} \xi_n^* u^{-n} \in \mathrm{End}(\mathcal{F})[[u,u^{-1}]] \\ \Gamma(u) &:= \sigma\circ X(u)\circ\sigma^{-1} \in \mathrm{End}(\mathcal{B})[[u,u^{-1}]] \\ \Gamma^*(u) &:= \sigma\circ X^*(u)\circ\sigma^{-1} \in \mathrm{End}(\mathcal{B})[[u,u^{-1}]] \end{aligned}}$$

<u>Theorem 1</u>: For any $m \in \mathbb{Z}$, the operators $\Gamma(u): \mathcal{B}^{(m)} \to \mathcal{B}^{(m+1)}$, $\Gamma^*(u): \mathcal{B}^{(m)} \to \mathcal{B}^{(m-1)}$
(their restrictions to graded components)
are explicitly given via the $\mathcal{A}$-action by:

$$\boxed{\begin{aligned} \Gamma(u) &= u^{m+1}\cdot z\cdot \exp\Big(\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)\exp\Big(-\sum_{j>0}\frac{a_j}{j}u^{-j}\Big) \\ \Gamma^*(u) &= u^{-m}\cdot z^{-1}\cdot \exp\Big(-\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)\exp\Big(\sum_{j>0}\frac{a_j}{j}u^{-j}\Big) \end{aligned}}$$

<u>Rmk 1</u>: (i) The factors $z$ and $z^{-1}$ are exactly in charge of $\mathcal{B}^{(m)} \to \mathcal{B}^{(m\pm1)}$

(ii) $\exp(A)$ is a formal expression $1+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots$

(iii) Note that $\{a_j\}_{j>0}$ (resp. $\{a_{-j}\}_{j>0}$) pairwise commute

(iv) For every $w \in \mathcal{B}^{(m)}$ each $u$-coeff. of the action of RHS on $w$ is well-defined!

<u>Rmk 2</u>: As $a_0$ acts by $m\cdot\mathrm{Id}$ on $\mathcal{B}^{(m)}$, we see that above formulas may be morally written as

$$\boxed{\Gamma(u) = uz\cdot :\exp\Big(\int a(u)\,du\Big):\ ,\quad \Gamma^*(u) = z^{-1}\cdot :\exp\Big(-\int a(u)\,du\Big):}$$

The proof of Theorem is based on the following result:

<u>Lemma 2</u>: For $j \in \mathbb{Z}$, we have $\boxed{[a_j, \Gamma(u)] = u^j\Gamma(u)\ ,\ [a_j, \Gamma^*(u)] = -u^j\Gamma^*(u)}$

- Applying the isom. $\sigma$, the first equality is equivalent to $[T^j, X(u)] = u^j X(u)$ (where $T$ is the shift operator from last time).

$$T^j = \sum_{i\in\mathbb{Z}} E_{i,i+j} \Rightarrow \hat{\varrho}(T^j) = \sum_i \hat{\varrho}(E_{i,i+j}) \overset{\text{P.1}}{=\!=} \sum_i \begin{cases} \xi_i\xi_{i+j}^* - 1, & \text{if } i=i+j\le 0 \\ \xi_i\xi_{i+j}^*, & \text{otherwise} \end{cases}$$

$$\overset{\text{on } \mathcal{F}}{\Longrightarrow} [\hat{\varrho}(T_j), X(u)] = \sum_{i\in\mathbb{Z}} [\xi_i\xi_{i+j}^*, X(u)] = \sum_{i,n\in\mathbb{Z}} [\xi_i\xi_{i+j}^*, \xi_n]u^n \ \textcircled{=}$$

<u>But</u>: $[\xi_i\xi_{i+j}^*, \xi_n] = \xi_i\xi_{i+j}^*\xi_n - \xi_n\xi_i\xi_{i+j}^* = \xi_i(-\xi_n\xi_{i+j}^* + \delta_{n,i+j}) - \xi_n\xi_i\xi_{i+j}^* = \delta_{n,i+j}\xi_i$

$$\textcircled{=} \sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} \delta_{n,i+j}\xi_i u^n = \sum_n \xi_{n-j}u^n = \underline{u^j X(u)}$$

- <u>Exercise</u>: Verify the second equality

## Proof of Theorem 1

▶ Define $\Gamma_+(u) := \exp\left(-\sum_{i>0} \frac{a_i}{i} u^{-i}\right) \in \mathrm{End}(\mathcal{B})[[u^{-1}]]$. Then, we have:

- $[a_j, \Gamma_+(u)] = 0 \quad \forall j \geq 0$ (as $\{a_{\geq 0}\}$ pairwise commute)
- $[a_j, \Gamma_+(u)] = u^j \Gamma_+(u) \quad \forall j < 0$ (it suffices to prove $[a_j, \exp(-\frac{a_{-j}}{-j} u^j)] = u^j \exp(-\frac{a_{-j}}{-j} u^j)$ which is easily checked just expanding $\exp(\ldots)$ formally or alternatively think of $a_j \leftrightarrow x_j$, $a_{-j} \leftrightarrow (-j)\partial/\partial x_j$)

As the constant term of $\Gamma_+(u)$ is 1, its inverse $\Gamma_+(u)^{-1}$ is well-defined and we get (from above):

- $[a_j, \Gamma_+(u)^{-1}] = 0$ if $j \geq 0$
- $[a_j, \Gamma_+(u)^{-1}] = -u^j \Gamma_+(u)^{-1}$ if $j < 0$

Finally, we define a linear map $\Delta(u) \colon \mathcal{B}^{(m)} \longrightarrow \mathcal{B}^{(m)}((u))$ via (same component!)

$$\boxed{\Delta(u) := \Gamma(u)\Gamma_+(u)^{-1} z^{-1}}$$

Combining the previous equalities with Lemma 2 and an obvious equality $[a_j, z] = \delta_{j,0} \cdot z$, we get:

$$\boxed{[a_j, \Delta(u)] = \begin{cases} u^j \cdot \Delta(u), & \text{if } j > 0 \\ 0, & \text{if } j \leq 0 \end{cases}}$$

As a consequence: (for any polynomial $P$)

$$\boxed{P(a_{-1}, a_{-2}, a_{-3}, \ldots)\, \Delta(u)\, \mathbb{1}_m = \Delta(u)\, P(a_{-1}, a_{-2}, a_{-3}, \ldots) \cdot \mathbb{1}_m}$$

(here $\mathbb{1}_m$ denotes h.wt. vector for $1 \in \mathcal{B}^{(m)}$)

Therefore, it suffices to compute $\Delta(u)\mathbb{1}_m \in \mathcal{B}^{(m)}((u))$, since $\mathcal{B}^{(m)}$ is generated by $\mathbb{1}_m$ over $\langle a_{-1}, a_{-2}, \ldots \rangle$.

On the other hand: $a_j \Delta(u) \mathbb{1}_m \overset{j>0}{=} \Delta(u) \underbrace{a_j \mathbb{1}_m}_{=0 \text{ for } j>0} + u^j \Delta(u) \mathbb{1}_m \implies \boxed{a_j(\Delta(u)\mathbb{1}_m) = u^j \cdot \Delta(u)\mathbb{1}_m \quad \forall j > 0}$

But: $a_j$ $(j>0)$ acts on Fock via $j\frac{\partial}{\partial x_j}$

$\implies \Delta(u)\mathbb{1}_m = F(u) \cdot \exp\left(\sum_{j>0} \frac{x_j}{j} u^j\right) = F(u) \cdot \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \mathbb{1}_m, \quad F(u) \in \mathbb{C}((u))$

$$\implies \boxed{\Gamma(u) = z \cdot F(u) \cdot \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)}$$

Hence, it remains to find the series $F(u)$. Since the isom. $\sigma$ identifies $\mathbb{1}_m \longleftrightarrow \psi_m$, we have

$\underbrace{\langle \mathbb{1}^*_{m+1}, \Gamma(u)\, \mathbb{1}_m \rangle}_{= F(u)} = \langle \psi^*_{m+1}, X(u)\psi_m \rangle = \langle \psi^*_{m+1}, \sum_n \xi_n u^n \psi_m \rangle = u^{m+1} \implies \boxed{F(u) = u^{m+1}}$

This completes the proof of the formula for $\Gamma(u)$ from Theorem 1.

Exercise: Prove the formula for $\Gamma^*(u)$.

■

**Corollary 1:** Let $\Gamma(u,v) := \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j}\right) \cdot \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j\right)$.

(a) $\rho\left(\sum_{i,j \in \mathbb{Z}} u^i v^{-j} E_{ij}\right) \underset{\substack{\text{on } \mathcal{B}^{(m)} \\ \text{or } \mathcal{F}^{(m)}}}{=\!=} \frac{(u/v)^m}{1 - v/u} \cdot \Gamma(u,v)$

(b) $\hat{\rho}\left(\sum_{i,j} u^i v^{-j} E_{ij}\right) \underset{\substack{\text{on } \mathcal{B}^{(m)} \\ \text{or } \mathcal{F}^{(m)}}}{=\!=} \frac{1}{1 - v/u} \left(\left(\frac{u}{v}\right)^m \Gamma(u,v) - 1\right)$

## Proof of Corollary 1

• (a) $\Rightarrow$ (b) as $\hat{\rho}(\sum_{i,j} u^i v^j E_{ij}) = \rho(\sum_{i,j} u^i v^j E_{ij}) - \underbrace{(\sum_{i\le 0} u^i v^{-i})}_{1/(1-\frac{v}{u})}\cdot \mathrm{Id}$

$\leftarrow$ expanded in non-negative powers of $v$.

• To prove (a), recall $\rho(E_{ij}) = \xi_i \xi_j^* \Rightarrow \rho(\sum_{i,j} u^i v^j E_{ij}) = X(u)\, X^*(v)$.

But, due to Theorem 1 we have:

$$X(u)\,X^*(v) = u^{(m-1)+1}\cdot z\cdot \exp\Big(\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)\exp\Big(-\sum_{j>0}\frac{a_j}{j}u^{-j}\Big)v^{-m}z^{-1}\exp\Big(-\sum_{j>0}\frac{a_{-j}}{j}v^j\Big)\exp\Big(\sum_{j>0}\frac{a_j}{j}v^{-j}\Big)$$

$$= \underbrace{\Big(\frac{u}{v}\Big)^m\cdot \exp\Big(\sum_{j>0}\frac{a_{-j}(u^j-v^j)}{j}\Big)\exp\Big(-\sum_{j>0}\frac{a_j(u^{-j}-v^{-j})}{j}\Big)}_{\Gamma(u,v)}\cdot G(u,v),$$

where $G(u,v)$ is the extra factor obtained by pulling $\exp(-\sum_{j>0}\frac{a_j}{j}u^{-j})$ to the right of $\exp(-\sum_{j>0}\frac{a_{-j}}{j}v^j)$. But as $[a_j, a_{-j'}] = j\cdot\delta_{jj'}$, we get:

$$G(u,v) = \exp\Big(\sum_{j>0}\frac{j}{j\cdot j}\cdot\frac{v^j}{u^j}\Big) = \exp(-\log(1-\tfrac{v}{u})) = \frac{1}{1-v/u}$$

■

Exercise: Prove this!

Thus, we provided the answer to Q2 from the beginning of today's class.

Let us now answer Q1!

First, we shall recall Schur polynomials. We start from

**Def 3**: For $k\in\mathbb{Z}_{\ge 0}$, define $S_k(x)\in\mathbb{C}[x_1,x_2,x_3,\ldots]$ via the generating series

$$\boxed{\sum_{k\ge 0} S_k(x)\cdot z^k = \exp\Big(\sum_{i\ge 1} x_i z^i\Big)}$$

(we treat $x$ as a collection $(x_1,x_2,x_3,\ldots)$)

(e.g. $S_0(x)=1$, $S_1(x)=x_1$, $S_2(x)=\frac{1}{2}x_1^2+x_2$)

They are closely related to complete symmetric functions $h_k(y)$, which are defined via

$$\boxed{h_k(y) = h_k(y_1,\ldots,y_N) = \sum_{\substack{p_1,\ldots,p_N\in\mathbb{Z}_{\ge 0}\\ p_1+\ldots+p_N=k}} y_1^{p_1}y_2^{p_2}\cdots y_N^{p_N}}$$

(here $y$ denotes a finite collection $(y_1,y_2,\ldots,y_N)$)

or alternatively they may be defined via the generating series

$$\boxed{\sum_{k\ge 0} h_k(y)\cdot z^k = \prod_{j=1}^N \frac{1}{1-z\cdot y_j}}$$

To relate $S_k(x)$ to $h_k(y)$ we should substitute $x_n = \frac{y_1^n+\ldots+y_N^n}{n}$ $\forall n\ge 1$.

**Lemma 3**: If $x_n = \frac{y_1^n+\ldots+y_N^n}{n}$ $\forall n$, then $S_k(x) = h_k(y)$.

► $\sum_{k\ge 0} S_k(x)\cdot z^k = \exp(\sum_{n\ge 1} x_n z^n) = \exp\Big(\sum_{n\ge 1}\frac{(y_1^n+\ldots+y_N^n)z^n}{n}\Big) = \exp\Big(-\sum_{j=1}^N \log(1-zy_j)\Big) = \prod_{j=1}^N\frac{1}{1-zy_j} = \sum_{k\ge 0} h_k(y)z^k$ ■

**Def 4**: To any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m)$, define (bosonic) Schur polynomial

$$\boxed{S_\lambda(x) := \det\begin{pmatrix} S_{\lambda_1}(x) & S_{\lambda_1+1}(x) & \dots & S_{\lambda_1+m-1}(x) \\ S_{\lambda_2-1}(x) & S_{\lambda_2}(x) & \dots & S_{\lambda_1+m-2}(x) \\ \dots & \dots & \dots & \dots \\ S_{\lambda_m-m+1}(x) & \dots & \dots & S_{\lambda_m}(x) \end{pmatrix} = \det\left(S_{\lambda_i+j-i}(x)\right)}$$

**Rmk 3**: (a) We set $S_k(x) = 0$ if $k<0$.

(b) Clearly $S_\lambda(x)$ does not change as we add $0$ to $\lambda$ at the end

(c) "Bosonic" – for the fact it is not symmetric in $x$.

(d) To get usual Schur polynomials substitute $x_n = \frac{y_1^n + \dots + y_N^n}{n}$ to get $S_\lambda(x) = \det\left(h_{\lambda_i+j-i}(y)\right)_{i,j=1}^N$ = $\lambda^{th}$ symmetric Schur pol-l of $\{y_j\}_{j=1}^N$

**Rmk 4**: Let $N \geq m$, set $\Lambda := (\lambda_1, \dots, \lambda_m, \underbrace{0, \dots, 0}_{N-m})$ – viewed as a weight of $GL_N$ or $\mathfrak{gl}_N$, and let $L_\Lambda$ be the irr. repr-n of $GL_N$ of highest weight $\Lambda$. Its character $\chi_\Lambda : \underset{GL_N}{\underset{\cap}{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}}} \to \mathbb{C}$ is defined via $\chi_\Lambda\begin{pmatrix} y_1 & 0 \\ 0 & y_N \end{pmatrix} := \mathrm{Tr}_{L_\Lambda}\begin{pmatrix} y_1 & 0 \\ 0 & y_N \end{pmatrix}$

Then, the following classical result emphasizes relevance of Schur polynomials to representation theory:

$$\boxed{\chi_\Lambda(y) = S_\lambda(x) \quad \text{for } x_n = \frac{\sum_{j=1}^N y_j^n}{n} \ \forall n}$$

Note: For $\lambda = (\lambda_1)$, this recovers Lemma 3.

**Theorem 2**: For any $i_0 > i_1 > i_2 > \dots$ s.t. $i_k = -k$ for $k \gg 0$, we have

$$\boxed{\sigma(\underbrace{v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots}_{\in \mathcal{F}^{(0)}}) = \underbrace{S_\lambda(x)}_{\in \mathcal{B}^{(0)}} \quad \text{for } \lambda = (i_0, i_1+1, i_2+2, \dots)}$$

**! WARNING**: We should modify our definition of $\mathcal{A} \curvearrowright F_m$ by saying $\begin{cases} a_j \mapsto \frac{\partial}{\partial x_j} \\ a_{-j} \mapsto j x_j \end{cases}, j>0$
(while in Lecture 2, we had $a_j \mapsto j\frac{\partial}{\partial x_j}$, $a_{-j} \mapsto x_j$)

Set $P(x) := \sigma(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots)$, so that we need to prove $P(x) = S_\lambda(x)$

Introduce formal variables $y_1, y_2, \dots$

- $\langle \mathbb{1}, e^{y_1 a_1 + y_2 a_2 + \dots} P(x) \rangle = \langle \mathbb{1}, e^{y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + \dots} P(x) \rangle = \langle \mathbb{1}, P(x+y) \rangle = P(y)$

On the other hand, recalling that $a_k \leftrightarrow T^k \in \mathfrak{a}_\infty$, we have:

- $\langle \mathbb{1}, e^{y_1 a_1 + y_2 a_2 + \dots} P(x) \rangle = \langle \psi_0, \underbrace{e^{y_1 T + y_2 T^2 + \dots}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots)}_{\text{Note: this is well-defined!}} \rangle \circleddash$

But: $\exp\left(\sum_{j \geq 1} y_j T^j\right) = \sum_{k \geq 0} S_k(y) T^k$

$\circleddash \langle \psi_0, \underbrace{\left(\sum_{k \geq 0} S_k(y) T^k\right)}_{\begin{pmatrix} 1 & S_1(y) & S_2(y) & \dots \\ & 1 & S_1(y) & \dots \\ 0 & & 1 & \dots \end{pmatrix}} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots) \rangle$ = coefficient of the basis elt $\psi_0$ in

ê $\begin{pmatrix} 1 & S_1(y) & S_2(y) & \dots \\ & 1 & S_1(y) & S_2(y) \\ 0 & & 1 & S_1(y) \end{pmatrix} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots)$

Combining two different computations of the contravariant pairing b/w $\mathbb{1}$ and $e^{y_1a_1+y_2a_2+\dots}P(x)$, we got

$$\boxed{\begin{array}{l} P(y) = \text{coefficient of the basis element } \psi_0 = v_0\wedge v_{-1}\wedge v_{-2}\wedge\dots \\ \text{in } \hat{\rho}\begin{pmatrix} 1 & S_1(y) & S_2(y) & \cdots \\ & 1 & S_1(y) & S_2(y) \\ & & 1 & S_1(y) & \cdots \\ 0 & & & \ddots & \cdots \end{pmatrix}(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\dots) \end{array}}$$

Extra argument is needed since this elt is not in $\sigma_\infty$
(But it is easy to see that only finitely many diagonals matter for computation of the matrix elts)

Recall that in the fin. dim. case, given a linear operator $A: V\to W$, whose action is determined in the distinguished basis $\{v_i\}_{i=1}^k$ of $V$, $\{w_j\}_{j=1}^l$ by an $l\times k$-matrix $A=(a_{ji})_{1\le j\le l}^{1\le i\le k}$, we have an induced linear operator $\Lambda^m A: \Lambda^m V\to\Lambda^m W$. Moreover, if we choose the basis of $\Lambda^m V$ to be $\{v_{i_1}\wedge v_{i_2}\wedge\dots\wedge v_{i_m} \mid 1\le i_1<i_2<\dots<i_m\le k\}$ and the basis of $\Lambda^m W$ to be $\{w_{j_1}\wedge w_{j_2}\wedge\dots\wedge w_{j_m} \mid j_1<j_2<\dots<j_m\}$, then the coefficient of $w_{j_1}\wedge\dots\wedge w_{j_m}$ in $\Lambda^m A(v_{i_1}\wedge v_{i_2}\wedge\dots\wedge v_{i_m})$ is given by the determinant $\det(A^{i_1i_2\dots i_m}_{j_1j_2\dots j_m})$, where $A^{i_1\dots i_m}_{j_1\dots j_m}$ is an $m\times m$-submatrix of $A$ formed by the intersection of rows $\# j_1, j_2,\dots j_m$ and columns $\# i_1, i_2,\dots, i_m$.

In the above setting, even though we are in the infinite dim. case, but starting from a certain moment action stabilizes, so that the coefficient of $v_0\wedge v_{-1}\wedge v_{-2}\wedge\dots$ in $\hat{\rho}\begin{pmatrix} 1 & S_1(y) & S_2(y) & \cdots \\ & 1 & S_1(y) & S_2(y) \\ & & 1 & S_1(y) \\ 0 & & & \ddots \end{pmatrix}(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\dots)$ is given by a determinant of $N\times N$ matrix

(here we choose $N$ so that $i_k=-k$ for $k\ge N$)

formed by intersecting columns $\# i_0, i_1,\dots i_{N-1}$ and rows $\# 0,-1,\dots -(N-1)$ of the matrix above, that is, the matrix

$$\begin{pmatrix} S_{\lambda_0}(y) & S_{\lambda_0+1}(y) & \cdots & S_{\lambda_0+(N-1)}(y) \\ S_{\lambda_1-1}(y) & S_{\lambda_1}(y) & \cdots & \\ \cdots & \cdots & \cdots & \cdots \\ S_{\lambda_N-(N+1)}(y) & \cdots & \cdots & S_{\lambda_N}(y) \end{pmatrix}, \text{ where as we agreed } \lambda_j = i_{j-1}+(j-1).$$

But: the determinant of this matrix is the Schur polynomial by def-n ∎

Corollary 2: Same arguments prove that the image of any elementary $\frac{\infty}{2}$-wedge of $\Lambda^{\frac{\infty}{2},m}V$ is given by

$$\boxed{\sigma(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\dots) = z^m\cdot S_{(i_0-m,\, i_1-m+1,\, i_2-m+2,\dots)}(x)}$$