— LECTURE 9 —

• Last time: Boson – Fermion Correspondence

$$\underset{\mathcal{A}\text{-module homom}}{\sigma}: \underset{\Lambda^{\frac{\infty}{2}}V}{\mathcal{F}} \xrightarrow{\ \sim\ } \underset{\mathbb{C}[z,z^{-1};x_1,x_2,x_3,\ldots]}{\mathcal{B}}$$

∘ We had wedging operators $\{\xi_n\}$, contracting operators $\{\xi_n^*\}$ acting naturally on $\mathcal{F}$

⇝ the corresponding quantum fields $X(u)=\sum_n \xi_n u^n$, $X^*(u)=\sum_n \xi_n^* u^{-n} \in \mathrm{End}(\mathcal{F})[[u,u^{-1}]]$

$\overset{\sigma}{⇝}$ the corresponding q. fields $\Gamma(u), \Gamma^*(u) \in \mathrm{End}(\mathcal{B})[[u,u^{-1}]]$ (better: $\Gamma(u)w, \Gamma^*(u)w \in \mathcal{B}((u))\ \forall w\in\mathcal{B}$)

Theorem 1 (last time): For any $m\in\mathbb{Z}$, the operators $\Gamma(u): \mathcal{B}^{(m)} \to \mathcal{B}^{(m+1)}$, $\Gamma^*(u): \mathcal{B}^{(m)} \to \mathcal{B}^{(m-1)}$ are explicitly given by the following vertex operator f-las in terms of $\mathcal{A}$-action:

$$\boxed{\begin{aligned}\Gamma(u)&=u^{m+1}\cdot z\cdot \exp\Big(\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)\exp\Big(-\sum_{j>0}\frac{a_j}{j}u^{-j}\Big) = \text{"}uz\cdot :\exp(\textstyle\int a(u)du):\text{"}\\ \Gamma^*(u)&=u^{-m}\cdot z^{-1}\cdot \exp\Big(-\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)\exp\Big(\sum_{j>0}\frac{a_j}{j}u^{-j}\Big) = \text{"}z^{-1}\cdot :\exp(-\textstyle\int a(u)du):\text{"}\end{aligned}}$$

As an immediate consequence of this result and the formula $\hat{\rho}(E_{ij})=\begin{cases}\xi_i\xi_j^*-1, & \text{if } i=j\le 0\\ \xi_i\xi_j^*, & \text{otherwise}\end{cases}$, we obtained:

Theorem 2 (last time):
$$\boxed{\sum_{i,j\in\mathbb{Z}} u^i v^{-j}\underset{\text{on } \mathcal{B}^{(m)} \text{ or } \mathcal{F}^{(m)}}{\hat{\rho}(E_{ij})}=\frac{1}{1-\frac{v}{u}}\cdot\Big(-1+\Big(\frac{u}{v}\Big)^m\cdot \exp\Big(\sum_{j>0}\frac{u^j-v^j}{j}a_{-j}\Big)\exp\Big(-\sum_{j>0}\frac{u^{-j}-v^{-j}}{j}a_j\Big)\Big)}$$

Those are one of the first examples (historically) illustrating importance of vertex operators...

(Technical) Remark 1: The following two formulas played a key technical role (and are used over and over again in similar vertex operator computations):

$$\boxed{\begin{aligned}&[a_{-n}, \exp(\tfrac{a_n}{n}u)] = -u\cdot\exp(\tfrac{a_n}{n}u)\\ &\exp(a_n\cdot u)\exp(a_{-n}v) = \exp(a_{-n}v)\exp(a_n u)\cdot\exp(nuv)\end{aligned}}$$
← Both equalities are equalities of the series in u (or u,v) with coefficients in $\mathrm{End}(\mathcal{B})$.

∘ We also introduced (bosonic) Schur polynomials $S_\lambda(x)\in\mathbb{C}[x_1,x_2,x_3,\ldots]$ (associated to any partition λ) s.t. upon specialization $x_n=\frac{y_1^n+\ldots+y_N^n}{n}\ \forall n\ge1$ (while $N\ge1$ is fixed from the beginning) we get the usual λ^{th} Schur polynomials $S_\lambda(y)$ (which are symmetric in $\{y_i\}$).

Theorem 3 (last time): For any $i_0>i_1>i_2>\ldots$ s.t. $i_k=-k$ for $k\gg0$, we have
$$\boxed{\sigma(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots)=S_{\lambda=(i_0,i_1+1,i_2+2,i_3+3,\ldots)}(x)}$$

Warning: For this formula to hold we had to redefine $\mathcal{A}\curvearrowright F_\mu$ via $\begin{cases}a_{-j}\mapsto jx_j\\ a_j\mapsto \partial/\partial x_j\end{cases}$ $(j>0)$, $a_0\mapsto\mu\,\mathrm{Id}$, $K\mapsto \mathrm{Id}$.

Remark 2: (a) More generally, if $i_0>i_1>i_2>\ldots$ and $i_k=m-k$ for $k\gg0$, then
$$\sigma(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots)=S_{(i_0-m,\,i_1-(m-1),\,i_2-(m-2),\ldots)}(x).$$

(b) The proof was almost tautological and solely based on the computations in $\mathcal{F}$-side (we evaluated $\langle\psi_0, e^{y_1a_1+y_2a_2+\ldots}(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots)\rangle$)

More details in class! → (c) Alternative proof is based on the verification that f-la of Theorem 3 defines an $\mathcal{A}$-homom (that is based on simple computations with determinants and expression of $S_\lambda(y)$ via det.-del.)

• Today: Application to the integrable systems.

One of the most important integrable PDEs is the KdV equation (Korteweg – de Vries) on $u=u(x,t)$:

$$\boxed{u_t = \frac{3}{2}u\cdot u_x + \frac{1}{4}u_{xxx}}$$

Remark 3: (a) Multiplying x, t, u by constants we may renormalize above constants $\frac{3}{2}, \frac{1}{4}$ to any other nonzero coeff-s.

(b) This equation describes motion of waves in shallow water.

(c) One of the most basic solutions (called "soliton") are:

$$u = \frac{2a^2}{\cosh^2(ax+a^3t)} \quad \text{for any } a\in\mathbb{C}^\times.$$

(d) Surprisingly, this eq-n has a lot of explicit solutions, unlike most of nonlinear PDE.

Another very important equation that shall appear in our discussion and which generalizes the KdV is the K.P equation (Kadomtsev – Petviashvili) on $u=u(x,y,t)$:

$$\boxed{u_{yy} = \left(u_t - \frac{3}{2}u\cdot u_x - \frac{1}{4}u_{xxx}\right)_x}$$

Goal: We shall construct a family of solutions of these equations using ∞-dim Lie algebras

Key tool: Infinite Grassmannian.

As a toy model, we start from the classical finite Grassmannian.

Setup: V – f.dim. vector space $/\mathbb{C}$ of dim n, $\{v_1,\dots,v_n\}$ – standard basis of $V\simeq\mathbb{C}^n$, integer $0\le k\le n$.

$\rightsquigarrow GL(V) \curvearrowright \Lambda^k V$ – irreducible repr-n with highest weight vector $v_1\wedge v_2\wedge\dots\wedge v_k$.

Def 1: Let $\Omega := GL(V)(v_1\wedge\dots\wedge v_k) \subset \Lambda^k V$

Lemma 1: Ω coincides with the set of decomposable nonzero k-wedges, i.e.

$$\boxed{\Omega = \{x\in\Lambda^k V \mid x = x_1\wedge\dots\wedge x_k \text{ for some linearly indep. } x_i\in V\}}$$

Obvious ∎

Def 2: The k-Grassmannian $Gr(k,V)$ of V is the set of all k-dim subspaces of V

$Gr(k,V)$ has a natural structure of a projective variety, which is based on the so-called Plücker embedding

$$\boxed{\begin{array}{ccc} Pl: Gr(k,V) & \longrightarrow & \mathbb{P}(\Lambda^k V) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \left(\begin{array}{c}k\text{-dim subspace } W \\ \text{with a basis } x_1,\dots,x_k\end{array}\right) & \longmapsto & \left(\begin{array}{c}\text{projection of } x_1\wedge\dots\wedge x_k\in\Lambda^k V\setminus\{0\} \\ \text{onto its projectivization}\end{array}\right)\end{array}}$$

Note: This is well-defined, i.e. is independent of the choice of a basis.

Exercise: Prove that Pl is injective.

We obviously get: $\mathrm{Im}(Pl) = \Omega/\mathbb{C}^\times$ $\Big\}$ $\Rightarrow$ $\boxed{Gr(k,V)\simeq \Omega/\mathbb{C}^\times}$

Completely analogously to Lecture 8, we may define

- wedging operator $\hat{v}_i : \Lambda^k V \to \Lambda^{k+1} V$
- contraction operator $\check{v}_i : \Lambda^k V \to \Lambda^{k-1} V$

(Note: $\forall v \in V$ $\hat{v} : \Lambda^k V \to \Lambda^{k+1} V$ $w \mapsto v \wedge w$
$\forall f \in V^*$ $\check{f} : \Lambda^k V \to \Lambda^{k-1} V$ via
$v_1 \wedge \ldots \wedge v_k \mapsto f(v_1) \cdot v_2 \wedge v_3 \wedge \ldots \wedge v_k - f(v_2) \cdot v_1 \wedge v_3 \wedge \ldots \wedge v_k + f(v_3) \cdot v_1 \wedge v_2 \wedge v_4 \wedge \ldots \wedge v_k - \ldots$)

Def 3: For any $0 \le k \le n$, define the linear operator $S : \Lambda^k V \otimes \Lambda^k V \to \Lambda^{k+1} V \otimes \Lambda^{k-1} V$ via
$$\boxed{S = \sum_{i=1}^n \hat{v}_i \otimes \check{v}_i}$$

Note: S is independent of the choice of a basis $\{v_i\}_{i=1}^n$ of V. ← Exercise: Prove this!

Theorem 4 (Plücker relations): For $\tau \in \Lambda^k V \setminus \{0\}$, we have
$$\boxed{\tau \in \Omega \iff S(\tau \otimes \tau) = 0}$$

($\Rightarrow$) As S is independent of the choice of a basis, we may assume $\tau = v_1 \wedge \ldots \wedge v_k$.
But then $\hat{v}_i(\tau) = 0$ if $1 \le i \le k$, while $\check{v}_i(\tau) = 0$ if $k < i \le n$ $\Rightarrow S(\tau \otimes \tau) = 0$.

($\Leftarrow$) Assume $S(\tau \otimes \tau) = 0$.
Define the subspace $E \subseteq V$ via $E = \{v \in V \mid \underbrace{\hat{v}\tau}_{v \wedge \tau} = 0\}$ (recall the wedging operator for any $v \in V$)
as well as the subspace $F \subseteq V^*$ via $F = \{f \in V^* \mid \check{f}\tau = 0\}$ (recall the contraction operator for any f).
As $\hat{v}\check{f} + \check{f}\hat{v} = f(v)\,\mathrm{Id}$, we immediately get
$$\boxed{E \subseteq F^{\perp} := \{v \in V \mid f(v) = 0 \ \forall f \in F\}}$$

Let $r := \dim E$, $s := \dim F^{\perp}$. Choose a basis $\{v_1, \ldots, v_n\}$ of V so that $\{v_1, \ldots, v_r\}$ is a basis of E, $\{v_1, \ldots, v_s\}$ – a basis of $F^{\perp}$. As $(F^{\perp})^{\perp} = F$, we see that $v_i^* \in F$ for $i > s$.

Hence: $\left.\begin{array}{l} \hat{v}_i \tau = 0 \text{ for } 1 \le i \le r \\ \check{v}_i^* \tau = 0 \text{ for } s < i \le n \end{array}\right\} \Rightarrow \boxed{S(\tau \otimes \tau) = \sum_{i=r+1}^{s} \hat{v}_i \tau \wedge \check{v}_i^* \tau}$ $\Bigg\} \Rightarrow$

But: The vectors $\{\hat{v}_i \tau \in \Lambda^{k+1} V\}_{i=r+1}^{s}$ are linearly independent
(indeed if $\exists c_i \in \mathbb{C} : \sum_{i=r+1}^s c_i \hat{v}_i \tau = 0 \Rightarrow (\sum_{i=r+1}^s c_i v_i)\tau = 0 \Rightarrow \sum_{i=r+1}^s c_i v_i \in E \Rightarrow$ all $c_i = 0$)
$\Rightarrow \check{v}_i^* \tau = 0$ for $r+1 \le i \le s \Rightarrow v_i^* \in F \ \forall r+1 \le i \le s \Rightarrow$ Contradiction, i.e. $r = s$, so that $E = F^{\perp}$.

Now we are ready to conclude that $\tau \in \Omega$. We shall write τ in the basis of $\Lambda^k V$:
$$\boxed{\tau = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} c_{i_1, i_2, \ldots, i_k} \cdot v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k}}$$

Exercise: (a) $\hat{v}_i \tau = 0$ for $1 \le i \le r \Rightarrow c_{i_1, \ldots, i_k} = 0$ if $\{1, \ldots, r\} \not\subseteq \{i_1, \ldots, i_k\}$
(b) $\check{v}_i^* \tau = 0$ for $r < i \le n \Rightarrow c_{i_1, \ldots, i_k} = 0$ if $\{i_1, \ldots, i_k\} \not\subseteq \{1, \ldots, r\}$

Due to this simple exercise, we see that $c_{i_1, \ldots, i_k} \ne 0$ iff $k = r$ and $i_p = p$ for $1 \le p \le k$
$\Rightarrow \tau$ is a nonzero multiple of $v_1 \wedge \ldots \wedge v_k \Rightarrow \tau \in \Omega$. ∎

One can make the above discussion coordinate-explicit.
Recall that $\{v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} \mid i_1 < i_2 < \ldots < i_k\}$ forms an explicit basis of $\Lambda^k V$.
Moreover the Plücker embedding may be described as follows:
given a k-dim subspace $W \subset V$ choose a basis $\{x_1, \ldots, x_k\}$ of $W \rightsquigarrow$ decomposing x_i in the basis $\{v_j\}_{j=1}^n$, we obtain an $n \times k$ matrix $A = (a_{ij})$. For any subset $I = \{i_1 < \ldots < i_k\} \subset \{1, \ldots, n\}$,

define the Plücker coordinate $P_I := \det(a_{ij})_{\substack{i\in I\\ 1\le j\le k}}$ – maximal $m\times m$ minor of A.

Then: $Pl(W)$ has coordinates $\{P_I\}_{|I|=k}$ in $\mathbb{P}(\Lambda^k V)$.

Note: $\boxed{x_1\wedge x_2\wedge\ldots\wedge x_k = \sum_{\substack{|I|=k\\ I=\{i_1<\ldots<i_k\}}} P_I\cdot v_{i_1}\wedge\ldots\wedge v_{i_k}}$

The following is a coordinate form of Theorem 4:

Corollary 1: For $\tau\in\Lambda^k V\setminus\{0\}$:

$$\boxed{\tau\in\Omega \iff \sum_{j\in J,\, j\notin I} (-1)^{\mu(j)}(-1)^{\nu(j)-1} P_{I\cup\{j\}} P_{J\setminus\{j\}} = 0 \text{ for any } I,J\subseteq\{1,\ldots,n\} \text{ with } |I|=k-1,\ |J|=k+1}$$

$\mu(j)$ equals the number of el-s of I which are smaller than j

$\nu(j)$ is equal to the number of j when all el-s of J are ordered in increasing order.

Now we are ready to move towards ∞-Grassmannian.

First, we shall replace $\Lambda^k V \ni v_{i_1}\wedge\ldots\wedge v_{i_k}$ with $\mathcal{F}^{(0)} = \Lambda^{\frac{\infty}{2},0}V \ni v_0\wedge v_{-1}\wedge v_{-2}\wedge\ldots = \psi_0$.

To define an ∞-counterpart of Ω, we need the following definition:

Def 4: (a) Let $M(\infty)$ denote the set of matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ such that all but finitely many terms among $\{a_{ij}-\delta_{ij}\}_{i,j}$ are ZERO (i.e. $M(\infty)=\mathrm{Id}+\mathfrak{gl}_\infty$)

(b) Define $GL(\infty)\subseteq M(\infty)$ as a subset of invertible elements

Exercise (Hwk 5): (a) $M(\infty)$ is a monoid under multiplication of matrices

(b) $GL(\infty)$ is a group under multiplication of matrices

(c) There is a natural monoid action $M(\infty)\curvearrowright \mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2},m}V$

$A(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots) = Av_{i_0}\wedge Av_{i_1}\wedge Av_{i_2}\wedge\ldots$

(d) There is a natural group action $GL(\infty)\curvearrowright\mathcal{F}^{(m)}$ given by the same f-la.

Def 5: Define $\Omega\subseteq\mathcal{F}^{(0)}$ via $\boxed{\Omega := GL(\infty)\psi_0}$

Lemma 2: For any $i_0>i_1>i_2>\ldots$ with $i_k=-k$ for $k\gg 0$, we have $v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots\in\Omega$

▶ There is a permutation $\sigma:\mathbb{Z}\xrightarrow{1\text{-to-}1}\mathbb{Z}$ s.t. $\sigma(k)=k$ for all but finitely many $k\in\mathbb{Z}$, and also $i_k=\sigma(-k)$ $\forall k\in\mathbb{Z}_{\ge 0}$. Then, σ may be viewed as an element of $GL(\infty)$ and $\sigma\psi_0 = v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots$ ■

Def 6: For any $m\in\mathbb{Z}$, define the linear operator $S:\mathcal{F}^{(m)}\otimes\mathcal{F}^{(m)}\to\mathcal{F}^{(m+1)}\otimes\mathcal{F}^{(m-1)}$ via

$$\boxed{S := \sum_{i\in\mathbb{Z}} \hat{v}_i\otimes\check{v}_i}$$

Note: For any $w\in\mathcal{F}^{(m)}\otimes\mathcal{F}^{(m)}$, the expression $S(w)$ is well-defined as only finitely many terms of $\{\hat{v}_i\otimes\check{v}_i(w)\}_{i\in\mathbb{Z}}$ are nonzero!

Theorem 5: For $\tau \in \mathcal{F}^{(0)} \setminus \{0\}$, we have
$$\boxed{\tau \in \Omega \iff S(\tau \otimes \tau) = 0}$$

The proof of this result is analogous to fin. dim. case (and actually can be reduced from it)

Exercise (Hwk 5): Prove Theorem 5.

Recalling that in fin. dim. case, we had $Gr(k,V) \simeq \Omega/\mathbb{C}^\times$, we are ready to give the key definition for today:

Def 7: The *(semi)infinite Grassmannian* Gr is defined via $\boxed{Gr := \Omega/\mathbb{C}^\times}$

Identifying $v_i \in V$ $(i \in \mathbb{Z})$ with $t^{-i} \in \mathbb{C}((t))$, we get the following down-to-earth interpretation of Gr:

$$\boxed{Gr = \left\{ \underset{\text{subspace}}{E \subseteq V} \;\middle|\; \begin{array}{l} t^k\mathbb{C}[[t]] \subseteq E \text{ for } k \gg 0 \\ \text{and } \dim(E/t^k\mathbb{C}[[t]]) = k \text{ for these } k \gg 0 \end{array} \right\}}$$

— LECTURE 10 —

Rmk 4: (a) If $t^k\mathbb{C}[[t]] \subseteq E$ and $\dim(E/t^k\mathbb{C}[[t]]) = k$, then $t^r\mathbb{C}[[t]] \subseteq E$ and $\dim(E/t^r\mathbb{C}[[t]]) = r$ $\forall r \geq k$.

(b) If $E \in Gr$, then $\exists k \gg 0$ s.t. $t^k\mathbb{C}[[t]] \subseteq E \subseteq t^{-k}\mathbb{C}[[t]] \Rightarrow E/t^k\mathbb{C}[[t]] \subseteq t^{-k}\mathbb{C}[[t]]/t^k\mathbb{C}[[t]] \simeq \mathbb{C}^{2k}$

(c) As an immediate corollary of (b), we see that
$$\boxed{Gr = \bigcup_{k \geq 1} Gr(k, 2k)}$$
! NOT a disjoint union, but rather a nested union!

Main Objective: Rewrite infinite Plücker relations of Theorem 5 using boson-fermion correspondence in terms of polynomials. In other words, we want to find a condition on $\tau \in \mathcal{B}^{(0)}$ to satisfy $S(\sigma^{-1}(\tau) \otimes \sigma^{-1}(\tau)) = 0$, so that $\sigma^{-1}(\tau) \in \Omega$ (here $\sigma: \mathcal{F}^{(0)} \xrightarrow{\sim} \mathcal{B}^{(0)}$)

Recalling the quantum fields $X(u) = \sum_{i \in \mathbb{Z}} \xi_i u^i = \sum_{i \in \mathbb{Z}} \hat{v}_i u^i$, $X^*(u) = \sum_{i \in \mathbb{Z}} \xi_i^* u^{-i} = \sum_{i \in \mathbb{Z}} \check{v}_i u^{-i}$, we see that
$$\boxed{S(\tau \otimes \tau) = 0 \iff \underbrace{CT_u}_{\text{constant term } = \text{ coeff. of } u^0}(X(u)\tau \otimes X^*(u)\tau) = 0}$$

Rmk 5: (a) $X(u)\tau, X^*(u)\tau \in \mathcal{F}((u)) \Rightarrow X(u)\tau \otimes X^*(u)\tau$ may be viewed as an element of $(\mathcal{F} \otimes \mathcal{F})((u))$

(b) For any algebra A and $\sum a_i u^i \in A((u))$, we set $CT_u(\sum a_i u^i) := a_0$.

Recalling that under the boson-fermion correspondence $\sigma: \mathcal{F} \xrightarrow{\sim} \mathcal{B}$ the quantum fields $X(u), X^*(u)$ on the fermionic side correspond to $\Gamma(u), \Gamma^*(u)$ on the bosonic side, we arrive at
$$\boxed{CT_u(\Gamma(u)\tau \otimes \Gamma^*(u)\tau) = 0 \quad \text{with } \tau \in \mathcal{B}^{(0)} = F_0 = \mathbb{C}[x_1, x_2, \ldots]}$$

To write down-to-earth this equality, we shall identify
$$\begin{array}{rcl} F_0 \otimes F_0 & \xrightarrow{\sim} & \mathbb{C}[x_1', x_1'', x_2', x_2'', x_3', x_3'', \ldots] \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ P \otimes Q & \longmapsto & P(x')Q(x'') \end{array}$$

Then, applying the explicit vertex operator formulas for $\Gamma(u), \Gamma^*(u)$, we may rewrite above as
$$\boxed{CT_u\left(u \cdot e^{\sum_{j>0} x_j' u^j} \cdot e^{-\sum_{j>0} \frac{1}{j}\frac{\partial}{\partial x_j'} u^{-j}} \cdot e^{-\sum_{j>0} x_j'' u^j} e^{\sum_{j>0} \frac{1}{j} \partial/\partial x_j'' \cdot u^{-j}} \tau(x_1', x_2', x_3', \ldots)\tau(x_1'', x_2'', x_3'', \ldots)\right) = 0}$$

! We use the updated action $\mathcal{A} \curvearrowright F$ over there (see Warning on p.1)

$$\Updownarrow$$

$$\boxed{CT_u\left(u \cdot \exp\left(\sum_{j>0}(x_j' - x_j'')u^j\right) \cdot \exp\left(\sum_{j>0}\left(\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''}\right)\frac{u^{-j}}{j}\right)\tau(x')\tau(x'')\right) = 0}$$