Theorem 5: For $\tau \in \mathcal{F}^{(0)} \setminus \{0\}$, we have

$$\boxed{\tau \in \Omega \iff S(\tau \otimes \tau) = 0}$$

The proof of this result is analogous to fin. dim. case (and actually can be reduced from it)

Exercise (Hwk 5): Prove Theorem 5.

Recalling that in fin. dim. case, we had $Gr(k,V) \simeq \Omega/\mathbb{C}^\times$, we are ready to give the key definition for today:

Def 7: The (semi)infinite Grassmannian Gr is defined via $\boxed{Gr := \Omega/\mathbb{C}^\times}$

Identifying $v_i \in V$ ($i \in \mathbb{Z}$) with $t^{-i} \in \mathbb{C}((t))$, we get the following down-to-earth interpretation of Gr:

$$\boxed{Gr = \left\{ \underset{\text{subspace}}{E \subseteq V} \;\middle|\; \begin{array}{l} t^k\mathbb{C}[[t]] \subseteq E \text{ for } k \gg 0 \\ \text{and } \dim(E/t^k\mathbb{C}[[t]]) = k \text{ for these } k \gg 0 \end{array} \right\}}$$

— LECTURE 10 —

Rmk 1: (a) If $t^k\mathbb{C}[[t]] \subseteq E$ and $\dim(E/t^k\mathbb{C}[[t]]) = k$, then $t^r\mathbb{C}[[t]] \subseteq E$ and $\dim(E/t^r\mathbb{C}[[t]]) = r$ $\forall r \geq k$.

(b) If $E \in Gr$, then $\exists k \gg 0$ s.t. $t^k\mathbb{C}[[t]] \subseteq E \subseteq t^{-k}\mathbb{C}[[t]] \Rightarrow E/t^k\mathbb{C}[[t]] \subseteq t^{-k}\mathbb{C}[[t]]/t^k\mathbb{C}[[t]] \simeq \mathbb{C}^{2k}$

(c) As an immediate corollary of (b), we see that

$$\boxed{Gr = \bigcup_{k \geq 1} Gr(k, 2k)}$$

! NOT a disjoint union, but rather a nested union!

Main Objective: Rewrite infinite Plücker relations of Theorem 5 from Lecture 9 using boson-fermion correspondence in terms of polynomials. In other words, we want to find a condition on $\tau \in \mathcal{B}^{(0)}$ to satisfy $S(\sigma^{-1}(\tau) \otimes \sigma^{-1}(\tau)) = 0$, so that $\sigma^{-1}(\tau) \in \Omega$ (here $\sigma: \mathcal{F}^{(0)} \xrightarrow{\sim} \mathcal{B}^{(0)}$)

Recalling the quantum fields $X(u) = \sum_{i\in\mathbb{Z}} \xi_i u^i = \sum_{i \in \mathbb{Z}} \hat{v}_i u^i$, $X^*(u) = \sum_{i \in \mathbb{Z}} \xi_i^* u^{-i} = \sum_{i\in\mathbb{Z}} \check{v}_i u^{-i}$, we see that

$$\boxed{S(\tau \otimes \tau) = 0 \iff CT_u(X(u)\tau \otimes X^*(u)\tau) = 0}$$

constant term = coeff. of u^0

Rmk 2: (a) $X(u)\tau, X^*(u)\tau \in \mathcal{F}((u)) \Rightarrow X(u)\tau \otimes X^*(u)\tau$ may be viewed as an element of $(\mathcal{F}\otimes\mathcal{F})((u))$

(b) For any algebra A and $\sum a_i u^i \in A((u))$, we set $CT_u(\sum a_i u^i) := a_0$.

Recalling that under the boson-fermion correspondence $\sigma: \mathcal{F} \xrightarrow{\sim} \mathcal{B}$ the quantum fields $X(u), X^*(u)$ on the fermionic side correspond to $\Gamma(u), \Gamma^*(u)$ on the bosonic side, we arrive at

$$\boxed{CT_u(\Gamma(u)\tau \otimes \Gamma^*(u)\tau) = 0 \quad \text{with } \tau \in \mathcal{B}^{(0)} = F_0 = \mathbb{C}[x_1, x_2, \ldots]}$$

To write down-to-earth this equality, we shall identify

$$F_0 \otimes F_0 \simeq \mathbb{C}[x_1', x_1'', x_2', x_2'', x_3', x_3'', \ldots]$$
$$P \otimes Q \mapsto P(x')Q(x'')$$

Then, applying the explicit vertex operator formulas for $\Gamma(u), \Gamma^*(u)$, we may rewrite above as

! We use the updated action $\mathcal{A} \curvearrowright F$ over there

$$\boxed{CT_u\left(u \cdot e^{\sum_{j>0} x_j' u^j} \cdot e^{-\sum_{j>0} \frac{1}{j}\frac{\partial}{\partial x_j'} u^{-j}} \cdot e^{-\sum_{j>0} x_j'' u^j} e^{\sum_{j>0} \frac{1}{j} \partial/\partial x_j'' \cdot u^{-j}} \tau(x_1', x_2', x_3', \ldots)\tau(x_1'', x_2'', x_3'', \ldots)\right) = 0}$$

$$\Updownarrow$$

$$\boxed{CT_u\left(u \cdot \exp\Big(\sum_{j>0}(x_j' - x_j'')u^j\Big) \cdot \exp\Big(\sum_{j>0}\Big(\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''}\Big)\frac{u^{-j}}{-j}\Big)\tau(x')\tau(x'')\right) = 0}$$

Corollary 1: $\tau \in \mathcal{B}^{(0)}$ satisfies $\sigma^{-1}(\tau) \in \Omega$ if and only if

$$\boxed{CT_u \left(u \cdot \exp\left(\sum_{j>0} (x'_j - x''_j) u^j\right) \cdot \exp\left(\sum_{j>0} \left(\partial/\partial x'_j - \partial/\partial x''_j\right) \frac{u^{-j}}{-j}\right) \tau(x')\tau(x'') \right) = 0}$$

We shall now simplify this equation by using the following change of variables:

$$\begin{cases} x' = x - y \\ x'' = x + y \end{cases}$$, i.e. we shall identify $\mathbb{C}[x'_1, x''_1, x'_2, x''_2, \dots]$ with $\mathbb{C}[x_1, y_1, x_2, y_2, \dots]$ via $x'_j = x_j - y_j,\ x''_j = x_j + y_j$

Note:

(a) $x' - x'' = -2y$, i.e. $x'_j - x''_j = -2y_j$

(b) $-\frac{\partial}{\partial x'} + \frac{\partial}{\partial x''} = \frac{\partial}{\partial y}$, i.e. $\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j} = -\frac{\partial}{\partial y_j}$

Hence, the above equality of Corollary 1 may be rewritten as:

$$\boxed{CT_u \left(u \cdot \exp\left(-2 \sum_{j>0} u^j y_j\right) \exp\left(\sum_{j>0} \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}\right) \tau(x-y)\tau(x+y) \right) = 0}$$

We shall now simplify this equation even further, but that will require the following notation:

Def 1: For any $P(x) \in \mathbb{C}[\![x_1, x_2, x_3, \dots]\!]$, $f(x), g(x) \in \mathbb{C}[x_1, x_2, x_3, \dots]$, define $A(P, f, g) \in \mathbb{C}[x_1, x_2, x_3, \dots]$:

$$\boxed{A(P, f, g) := \left(P\left(\tfrac{\partial}{\partial z}\right)\left(f(x-z)g(x+z)\right)\right)_{|z=0}}$$

Rmk 3: (a) $\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \dots\right)$

(b) For $h(x,z) \in \mathbb{C}[x_1, z_1, x_2, z_2, \dots]$, $h_{|z=0}$ denotes $h(x, 0)$.

(c) The right-hand side above is well-defined.

Lemma 1: (a) If $P_-(x) := P(-x)$, then $A(P, f, g) = A(P_-, g, f)$

(b) If $P \in \mathbb{C}[x_1, x_2, \dots]$ is odd, then $A(P, f, f) = 0 \quad \forall f \in \mathbb{C}[x_1, x_2, \dots]$

▸ (a) Obvious

(b) Follows from (a) ∎

Theorem 1 (Hirota bilinear relations): For $\tau \in \mathcal{B}^{(0)} \setminus \{0\}$, we have $\sigma^{-1}(\tau) \in \Omega$ if and only if

$$(*) \quad \boxed{A\left(\sum_{j=0}^{\infty} S_j(-2y) S_{j+1}(\tilde{x}) \exp\left(\sum_{s>0} y_s x_s\right), \tau, \tau\right) = 0}$$, where we set $\tilde{x}_1 = x_1,\ \tilde{x}_2 = \frac{x_2}{2},\ \tilde{x}_3 = \frac{x_3}{3}, \dots$

Rmk 4: (a) Here, we view $P = \sum_{j\geq 0} S_j(-2y) S_{j+1}(\tilde{x}) e^{\sum_{s>0} y_s x_s}$ as an element of $(\mathbb{C}[\![y_1, y_2, \dots]\!])[\![x_1, x_2, x_3, \dots]\!]$

(b) Likewise, $\tau \in \mathbb{C}[x_1, x_2, x_3, \dots]$ shall be also viewed as an elt of $(\mathbb{C}[\![y_1, y_2, \dots]\!])[x_1, x_2, x_3, \dots]$

Before presenting the proof, let us recall that by the very definition of $S_j(\cdot)$, we have

$$\boxed{\begin{aligned} &\sum_{j\geq 0} S_j(-2y) u^j = \exp\left(-\sum_{j\geq 1} 2y_j u^j\right) \\ &\sum_{j\geq 0} S_j(\tilde{\partial}_y) u^{-j} = \exp\left(\sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}\right) \end{aligned}}$$ ← here $\tilde{\partial}_y = \left(\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, \dots\right)$

Proof of Theorem 1

$$CT_u\left(u\cdot \exp\Big(-\sum_{j\geq 1} 2y_j u^j\Big)\cdot \exp\Big(\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\Big)\tau(x+y)\tau(x-y)\right) = \quad \text{introduce new } t=(t_1,t_2,t_3,\dots)$$

$$CT_u\left(u\cdot \exp\Big(-\sum_{j\geq 1} 2y_j u^j\Big)\exp\Big(\sum_{j\geq 1}\frac{1}{j}\frac{\partial}{\partial y_j}u^{-j}\Big)\tau(x+y+t)\,\tau(x-y-t)\right)\Big|_{t=0} =$$

$$CT_u\left(u\cdot\Big(\sum_{j\geq 0} S_j(-2y)u^j\Big)\Big(\sum_{j\geq 0} S_j(\tilde{\partial}_y)u^{-j}\Big)\tau(x+y+t)\,\tau(x-y-t)\right)\Big|_{t=0} =$$

$$\Big(\sum_{j\geq 0} S_j(-2y)\,S_{j+1}(\tilde{\partial}_y)\,\tau(x+y+t)\,\tau(x-y-t)\Big)\Big|_{t=0} =$$

But: Clearly $S_{j+1}(\tilde{\partial}_y)\,\tau(x+y+t)\tau(x-y-t) = S_{j+1}(\tilde{\partial}_t)\,\tau(x+y+t)\tau(x-y-t)$

$$= \Big(\sum_{j\geq 0} S_j(-2y)\,S_{j+1}(\tilde{\partial}_t)\underbrace{\tau(x+y+t)\,\tau(x-y-t)}_{\exp(\sum_{s\geq 1} y_s\frac{\partial}{\partial t_s})\tau(x+t)\tau(x-t) \text{ by Taylor decomp.}}\Big)\Big|_{t=0} =$$

$$\Big(\sum_{j\geq 0} S_j(-2y)\,S_{j+1}(\tilde{\partial}_t)\,\exp\Big(\sum_{s\geq 1} y_s\frac{\partial}{\partial t_s}\Big)\big(\tau(x+t)\,\tau(x-t)\big)\Big)\Big|_{t=0} \;\underset{\text{following conventions of Remark 4}}{\overset{\text{by Def 1}}{=}}$$

$$\boxed{A\Big(\sum_{j\geq 0} S_j(-2y)\,S_{j+1}(\tilde{x})\exp\Big(\sum_{s>0} y_s x_s\Big),\ \tau,\ \tau\Big)}$$

As $\sigma^{-1}(\tau)\in\Omega \iff S(\sigma^{-1}(\tau)\otimes\sigma^{-1}(\tau))=0$, we finally see:

$$\sigma^{-1}(\tau)\in\Omega \iff A\Big(\sum_{j\geq 0} S_j(-2y)\,S_{j+1}(\tilde{x})\exp\Big(\sum_{s>0} y_s x_s\Big),\ \tau,\ \tau\Big)=0$$

Note that the equality $(*)$ (see p.2) is actually a family of equalities (one for each monomial in y-variables)! In other words, for any $\gamma=(\gamma_1,\gamma_2,\dots)$ (almost all of which are 0), consider the summand of $(*)$ with y-variables appearing in the form $y_1^{\gamma_1}y_2^{\gamma_2}\cdots$

- $\gamma_1=\gamma_2=\dots=0$, i.e. degree 0 term in y.

  Then $(*)$ just gives $A(\underbrace{S_1(\tilde{x})}_{x_1},\tau,\tau)=0$, which automatically holds due to Lemma 1(b) and the fact that x_1 - odd.

- $\gamma_r=1$, $\gamma_s=0$ for $s\neq r$, i.e. degree 1 terms in y.

  The coefficient of the monomial y_r in $\sum_{j\geq 0} S_j(-2y)S_{j+1}(\tilde{x})\exp(\sum_{s>0} y_s x_s)$ equals $x_1x_r-2S_{r+1}(\tilde{x})$. Therefore, $(*)$ implies in that case

$$\boxed{A(x_1x_r-2S_{r+1}(\tilde{x}),\tau,\tau)=0 \quad \forall\, r\geq 1}$$

Def 2: The system of above equations usually goes under the name "KP hierarchy".

Let us see what the first of these equations look like. We define $T_r(x)=x_1x_r-2S_{r+1}(\tilde{x})$

E.g. $T_1(x)=-x_2$ (as $S_2(x)=\frac{x_1^2}{2}+x_2$) $\Rightarrow T_1(x)$ - odd

$T_2(x)=-\frac{x_1^3}{3}-\frac{2x_3}{3}$ (as $S_3(x)=\frac{x_1^3}{6}+x_1x_2+x_3$) $\Rightarrow T_2(x)$ - odd

$\overset{\text{Lemma 1(b)}}{\Longrightarrow}$ the first two eq-s in the KP hierarchy are tautological!

$T_3(x)=\frac{x_1x_3}{3}-\frac{x_4}{2}-\frac{x_2^2}{4}-\frac{x_1^4}{12}-\frac{x_1^2x_2}{2}$ (as $S_4(x)=\frac{x_1^4}{24}+\frac{x_1^2x_2}{2}+\frac{x_2^2}{2}+x_1x_3+x_4$)

Note that $T_3(x)$ is not odd, and ignoring its 2 odd summands, we see that the first nontrivial equation of the above KP hierarchy reads

$$\boxed{A\left(\frac{x_1x_3}{3}-\frac{x_2^2}{4}-\frac{x_1^4}{12},\tau,\tau\right)=0}$$

$$\Downarrow$$

$$\boxed{\left(\left(\frac{1}{3}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3}-\left(\frac{\partial}{\partial z_2}\right)^2\cdot\frac{1}{4}-\left(\frac{\partial}{\partial z_1}\right)^4\cdot\frac{1}{12}\right)(\tau(x-z)\tau(x+z))\right)_{|z=0}=0}$$

$$\Downarrow$$

$$\boxed{\left((\partial_{z_1}^4+3\partial_{z_2}^2-4\partial_{z_1}\partial_{z_3})\tau(x-z)\tau(x+z)\right)_{|z=0}=0} \quad (**)$$

To transform this into PDE, we make the substitution: $x_1=x$, $x_2=y$, $x_3=t$, $x_m=c_m$ for $m>3$.
Set

$$\boxed{u:=2\partial_x^2\log\tau}$$

Prop 1: τ satisfies $(**)$ iff u satisfies the KP eq-n $\boxed{\frac{3}{4}\partial_y^2u=\partial_x\left(\partial_t u-\frac{3}{2}u\cdot\partial_x u-\frac{1}{4}\partial_x^3u\right)}$

renormalized version of that eq-n from Lecture 9.

Exercise (Hwk 5): Prove this result!

Corollary 2: Any element of Gr gives rise to a solution of the KP equation

As we just checked $\tau\in\Omega\Rightarrow(**)\Rightarrow$ KP eq-n for $u=2\partial_x^2\log\tau$. Moreover u for τ & $\overset{\mathbb{C}^*}{c}\cdot\tau$ do coincide
$Gr=\Omega/\mathbb{C}^\times\Rightarrow$ every point of Gr provides a solution of KP eq-n ∎

Corollary 3: For any partition λ, $u=2\partial_x^2\log S_\lambda(x,y,t,c_4,c_5,\dots)$ is a solution of the KP eq-n (and actually the entire KP hierarchy), where $c_4,c_5,\dots$ are treated as constants.

Follows from Corollary 2 together with $\sigma^{-1}(S_\lambda(x))=v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\dots$ (for corresponding $i_0,i_1,i_2,\dots$) and the observation from Lecture 9 that all $v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\dots\in\Omega$ ∎

Next, we shall construct other solutions of the KP hierarchy.
Recall the quantum field $\Gamma(u,v)\in(\mathrm{End}(\mathcal{B}^{(0)}))[[u,u^{-1},v,v^{-1}]]$ given by

$$\boxed{\Gamma(u,v)=\exp\left(\sum_{j>0}\frac{u^j-v^j}{j}a_{-j}\right)\exp\left(-\sum_{j>0}\frac{u^{-j}-v^{-j}}{j}a_j\right)}$$

which was used in Lecture 8 (Corollary 1) to define the action $gl_\infty\curvearrowright\mathcal{B}$.
Let us rewrite $\Gamma(u,v)$ in an alternative form. Recall the vertex operator expressions for $\Gamma(u)$, $\Gamma^*(u)$ of Lecture 8:

$$\Gamma^*(v)=z^{-1}\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}v^j\right)\exp\left(\sum_{j>0}\frac{a_j}{j}v^{-j}\right) \text{ on } \mathcal{B}^{(0)}$$

$$\Gamma(u)=z\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(-\sum_{j>0}\frac{a_j}{j}u^{-j}\right) \text{ on } \mathcal{B}^{(-1)}$$

Hence, we may express $\Gamma(u,v)$ as the following normally ordered expression:

$$\boxed{\Gamma(u,v)=\,:\Gamma(u)\Gamma^*(v):}$$

Let me now formulate the last key result for today, while its proof will require some technical discussions.

Theorem 2: If $\tau\in\Omega$, then $(1+a\Gamma(u,v))\tau\in\Omega_{u,v}$ $\forall a\in\mathbb{C}$, where

$$\boxed{\Omega_{u,v}=\{\tau\in\mathcal{B}^{(0)}((u,v))\mid S(\tau\otimes\tau)=0\}}$$

Rmk 5: (1) S is viewed as a map $(\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)})((u,v))\to(\mathcal{B}^{(1)}\otimes\mathcal{B}^{(-1)})((u,v))$ here

(2) $\tau\otimes\tau$ is also viewed as an elt of $\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}\subseteq\mathcal{B}^{(0)}((u,v))\otimes\mathcal{B}^{(0)}((u,v))\simeq(\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)})((u,v))$

Corollary 4: For any $a_1,\ldots,a_n\in\mathbb{C}$, we have

$$\boxed{(1+a_1\Gamma(u_1,v_1))(1+a_2\Gamma(u_2,v_2))\ldots(1+a_n\Gamma(u_n,v_n))\mathbb{1}\in\Omega_{u_1,v_1,\ldots,u_n,v_n}}$$

Combining this with Proposition 1, we finally get:

Proposition 2: For any $a_1,\ldots,a_n\in\mathbb{C}$, set $\tau=(1+a_1\Gamma(u_1,v_1))\ldots(1+a_n\Gamma(u_n,v_n))\mathbb{1}$. Then, $u=2\frac{\partial^2}{\partial x_1^2}(\log\tau)$ is a converged series and is a solution ("n-soliton" solution) of the KP hierarchy.

Example: For $n=1$, we get $\tau=(1+a\Gamma(u,v))\mathbb{1}=1+a\cdot e^{(u-v)x+(u^2-v^2)y+(u^3-v^3)t+\sum_{n\geq4}(u^n-v^n)c_n}$.

Absorbing $e^{\sum_{n\geq4}(u^n-v^n)c_n}$ into the constant a, we may write

$$\boxed{\tau=1+a\cdot e^{(u-v)x+(u^2-v^2)y+(u^3-v^3)t}=1+e^{(u-v)x+(u^2-v^2)y+(u^3-v^3)t+c}}$$

$$\Downarrow$$

$$u(x,y,t)=2\partial_x^2\log\tau=\frac{2(u-v)^2e^{(u-v)x+(u^2-v^2)y+(u^3-v^3)t+c}}{(1+e^{\cdots})^2}=\boxed{\frac{(u-v)^2}{2}\cdot\frac{1}{\cosh^2\frac{1}{2}((u-v)x+(u^2-v^2)y+(u^3-v^3)t+c)}}$$

To make this function independent of y, set $v=-u$, so that

$$\boxed{u(x,t)=\frac{2u^2}{\cosh^2(ux+u^3t+c/2)}}$$

Setting $c=0$, we recover exactly the soliton solution of KdV:

$$\boxed{u(x,t)=\frac{2u^2}{\cosh^2(ux+u^3t)}}$$

It remains to prove Theorem 2.

As we will see "$\Gamma(u,v)^2=0$" in appropriate sense, so that "$1+a\Gamma(u,v)=e^{a\Gamma(u,v)}$".

On the other hand, S commutes with $G(\infty)$-action, hence, S commutes with "$e^{a\Gamma(u,v)}$", implying Theorem 2.

To make the above argument rigorous, we start from the following Lemma:

Lemma 2:
$$\Gamma(u)\Gamma(v)=\frac{u-v}{u}:\Gamma(u)\Gamma(v):$$
$$\Gamma(u)\Gamma^*(v)=\frac{u}{u-v}:\Gamma(u)\Gamma^*(v):$$
$$\Gamma^*(u)\Gamma(v)=\frac{u}{u-v}:\Gamma^*(u)\Gamma(v):$$
$$\Gamma^*(u)\Gamma^*(v)=\frac{u-v}{u}:\Gamma^*(u)\Gamma^*(v):$$

$\ulcorner$ Straightforward ■

If we use $\Gamma_+(u), \Gamma_-(u)$ to denote $\Gamma(u), \Gamma^*(u)$, resp., then we obtain:

<u>Corollary 5</u>: $\boxed{\Gamma_{\varepsilon_1}(u_1)\dots\Gamma_{\varepsilon_n}(u_n) = \prod_{i<j}\left(\frac{u_i-u_j}{u_i}\right)^{\varepsilon_i\varepsilon_j}\cdot :\Gamma_{\varepsilon_1}(u_1)\dots\Gamma_{\varepsilon_n}(u_n):}$ (with the series expanded in $|u_1|>|u_2|>\dots>|u_n|$)

As the matrix coefficients $\langle w_1, :\Gamma_{\varepsilon_1}(u_1)\dots\Gamma_{\varepsilon_n}(u_n): w_2\rangle$ are always Laurent polynomials for any $w_1, w_2 \in \mathcal{B}^{(0)}$, we get to the following important result:

<u>Corollary 6</u>: All matrix coeff-s of $\Gamma_{\varepsilon_1}(u_1)\dots\Gamma_{\varepsilon_n}(u_n)$ are series which converge to rational functions of the form $\boxed{\prod_{i<j}\left(1-\frac{u_j}{u_i}\right)^{\varepsilon_i\varepsilon_j}\cdot P \text{ with } P\in\mathbb{C}[u_1^{\pm1},\dots,u_n^{\pm1}]}$.

In view of this observation, we get:

<u>Corollary 7</u>: $\boxed{\Gamma(u',v')\Gamma(u,v) = \frac{(u'-u)(v'-v)}{(u'-v)(v'-u)} :\Gamma'(u',v')\Gamma(u,v):}$

$$\left.\begin{aligned}\Gamma_+(u')\Gamma_-(v') &= \frac{u'}{u'-v'} :\Gamma_+(u')\Gamma_-(v'): = \frac{u'}{u'-v'}\Gamma(u',v')\\ \Gamma_+(u)\Gamma_-(v) &= \frac{u}{u-v} :\Gamma_+(u)\Gamma_-(v): = \frac{u}{u-v}\Gamma(u,v)\end{aligned}\right\} \Rightarrow \Gamma_+(u')\Gamma_-(v')\Gamma_+(u)\Gamma_-(v) = \frac{u'}{u'-v'}\cdot\frac{u}{u-v}\cdot\Gamma(u',v')\Gamma(u,v)$$

<u>But</u>, by Corollary 6: $\Gamma_+(u')\Gamma_-(v')\Gamma_+(u)\Gamma_-(v) = \frac{u'}{u'-v'}\cdot\frac{u'-u}{u'}\cdot\frac{u'}{u'-v}\cdot\frac{v'}{v'-u}\cdot\frac{v'-v}{v'}\cdot\frac{u}{u-v} :\Gamma_+(u')\Gamma_-(v')\Gamma_+(u)\Gamma_-(v):$

Finally, as all matrix coeff-s are rational f-s and the latter do not have zero divisors, we get

$$\Gamma'(u',v')\Gamma(u,v) = \frac{(u'-u)(v'-v)}{(u'-v)(v'-u)} :\Gamma_+(u')\Gamma_-(v')\Gamma_+(u)\Gamma_-(v): = \frac{(u'-u)(v'-v)}{(u'-v)(v'-u)} :\Gamma'(u',v')\Gamma(u,v):$$

∎

As an immediate consequence, we get

<u>Corollary 8</u>: If $u\neq v$, then $\boxed{\lim_{\substack{u'\to u\\ v'\to v}}\Gamma(u',v')\Gamma(u,v)=0}$ (i.e. all matrix coeff-s are ZERO!).

<u>Note</u>: This is a rigorous formulation of the aforementioned equality "$(\Gamma(u,v))^2=0$".

Now we are ready to prove Theorem 2.

<u>(Proof of Theorem 2)</u>

$$S((1+a\Gamma(u,v))\tau\otimes(1+a\Gamma(u,v))\tau) = \underbrace{S(\tau\otimes\tau)}_{=0} + a\underbrace{S(\Gamma(u,v)\tau\otimes\tau+\tau\otimes\Gamma(u,v)\tau)}_{=(\Gamma(u,v)\otimes1+1\otimes\Gamma(u,v))S(\tau\otimes\tau)=0} + a^2S(\Gamma(u,v)\tau\otimes\Gamma(u,v)\tau)$$

<u>But</u>: $S(\Gamma(u,v)\tau\otimes\Gamma(u,v)\tau) = \lim_{\substack{u'\to u\\ v'\to v}}\frac{1}{2}S(\Gamma(u,v)\tau\otimes\Gamma(u',v')\tau+\Gamma(u',v')\tau\otimes\Gamma(u,v)\tau)$

$$= \lim_{\substack{u'\to u\\ v'\to v}}\frac{1}{2}S((\Gamma(u',v')\otimes1+1\otimes\Gamma(u',v'))(\Gamma(u,v)\otimes1+1\otimes\Gamma(u,v))(\tau\otimes\tau)) -$$

$$- \lim_{\substack{u'\to u\\ v'\to v}}\frac{1}{2}S\Big(\big(\underbrace{\Gamma(u',v')\Gamma(u,v)}_{\to0 \text{ by Corollary } 8}\otimes1 + 1\otimes\underbrace{\Gamma(u',v')\Gamma(u,v)}_{\to0 \text{ by Corollary } 8}\big)(\tau\otimes\tau)\Big) = 0,$$

where again we used $S(x\otimes1+1\otimes x) = (x\otimes1+1\otimes x)S \;\; \forall x\in gl_\infty$

∎