— LECTURE 11 —

Today: Irreducibility of Virasoro Verma modules.

Recall: For any $\lambda=(c,h)\in\mathbb{C}^2$, we have the corresponding Verma module $M_\lambda^+=M_\lambda=M_{c,h}$ over the Virasoro, we shall use v_λ to denote its h.wt. vector.

According to Lectures 3-4, M_λ has a unique contravariant form $(\cdot,\cdot)$ s.t. $(v_\lambda,v_\lambda)=1$.
means $(L_n v,w)=(v,L_{-n}w)\ \forall n\in\mathbb{Z}$

This form is symmetric and is called the Shapovalov form.
(recall that it was obtained from the invariant bilinear form $M_\lambda^+\times M_{-\lambda}^-\to\mathbb{C}$ using involution ω mapping $L_n\mapsto -L_{-n}$)

Rmk 1: If $\lambda\in\mathbb{R}^2$, then as we saw in Lectures 4-5, Verma module M_λ has a Hermitian form $\langle\cdot,\cdot\rangle$, which satisfies the same defining properties: $\langle v_\lambda,v_\lambda\rangle=1$ and $\langle L_n v,w\rangle=\langle v,L_{-n}w\rangle$.

Also, the Verma module $M_{c,h}$ is naturally $\mathbb{Z}_{\le 0}$-graded, where we identify $M_{c,h}\underset{\text{graded v.spaces}}{\simeq}U(\mathfrak{n}_-)$, $\mathfrak{n}_-=\bigoplus_{n<0}\mathbb{C}\cdot L_n$. The degree components of $M_{c,h}=\bigoplus_{n>0}M_{c,h}[-n]$ are orthogonal to each other.

Moreover, for each $n>0$, we introduced the polynomial $\det_n(c,h)$, defined via

$\boxed{\det_n(c,h)=\det\left((x_I v_\lambda, x_J v_\lambda)_{I,J}\right)}$ where $\{x_I\}$ denotes the basis of $M_{c,h}[-n]$.

↑ it is uniquely determined up to a nonzero constant factor $\Rightarrow$ condition "$\det_n(c,h)=0$" is invariant!

According to Lecture 4, we have:

Lemma 1: (a) $M_{c,h}$ is irreducible iff $\det_n(c,h)\neq 0\ \forall n>0$

(b) $\det_n(c,h)=0\iff M_{c,h}$ contains a nonzero homogeneous singular vector of degree $\geq -n$.

Corollary 1: If $\det_n(c,h)=0$, then $\det_{n+1}(c,h)=0$ ($\Rightarrow \det_{>n}(c,h)=0$).

Example 1 (due to Lecture 4): $\det_1(c,h)=-2h$, $\det_2(c,h)=-4h\left((2h+1)(4h+\frac{c}{2})-9h\right)\Rightarrow \det_2 \vdots \det_1$.

Rmk 2: For a Hermitian form $\langle\cdot,\cdot\rangle$, the condition "$\det_n(c,h)>0$" is invariant of the choice of a basis! (as changing basis, we multiply by $|\det(?)|^2$)

Lemma 2: If $M_{c,h}$ is unitary, then $\det_n(c,h)>0$ $\forall n$ (where $\det_n(c,h)$ is defined using Hermitian form)

► Just b/c positive-definite Hermitian matrix has $\det\in\mathbb{R}_{>0}$ ∎

Theorem 1: Fix $c\in\mathbb{C}$. Then $\boxed{\det_m(c,h)=\underline{K_m}\cdot h^{\sum_{r,s\geq 1}^{rs\leq m}p(m-rs)}+(\text{lower order terms in } h)}$
nonzero constant explicitly given by
$K_m=\prod_{r,s\geq1}^{rs\leq m}\left((2r)^s\cdot s!\right)^{m(r,s)}$, $m(r,s):=p(m-rs)-p(m-r(s+1))$

$p(n)$ ‖ number of size n partitions

► Following our proof of [Lecture 4, Theorem 1], it is clear that the leading (in h) term of $\det_m$ comes from the contributions on the diagonal. On diagonal we have the following entries:

$\boxed{(L_{-r}^{k_r}\cdots L_{-1}^{k_1}v_\lambda,\ L_{-r}^{k_r}\cdots L_{-1}^{k_1}v_\lambda)=(v_\lambda,\ L_1^{k_1}\cdots L_r^{k_r}L_{-r}^{k_r}\cdots L_{-1}^{k_1}v_\lambda)\quad \text{with } k_1+2k_2+\ldots+rk_r=m}$

Moreover, $\{L_{-r}^{k_r}\cdots L_{-1}^{k_1}v_\lambda\mid k_1+2k_2+\ldots+rk_r=m\}$ form a basis of $M_{c,h}[-m]$.

Note that the leading power in h of $(v_\lambda, L_1^{k_1}\ldots L_r^{k_r}L_{-r}^{k_r}\ldots L_{-1}^{k_1}v_\lambda)$ equals $(k_1+\ldots+k_r)$ $\left(\text{note: the coefficient of it } \prod_{j=1}^{r}(2j)^{k_j}\cdot k_j!\right)$

So: The leading power in h of $\det_m(c,h)$ equals

$\boxed{\sum_{\mu\vdash m}\sum_i k_i(\mu)}$ where we encode $(k_1,\ldots,k_r)$ s.t. $\sum_{j=1}^r jk_j=m$ by a partition μ of m consisting of k_j terms equal to j ($\forall j$); $k_j(\mu)=\#\{\text{terms of } \mu \text{ equal to } j\}$

(Continuation of the proof of Theorem 1)

It remains to prove the equality $\sum_{\mu\vdash m}\sum_i k_i(\mu) \overset{?}{=} \sum_{\substack{r,s\geq 1\\ rs\leq m}} p(m-rs)$

Recall the above notation $m(r,s)=p(m-rs)-p(m-r(s+1))$ – it equals the number of partitions $\mu\vdash m$ in which r occurs exactly s times

This combinatorial meaning implies that $\forall r$: $\underbrace{\sum_s s\,m(r,s)}_{\text{equals the total number of occeurence of } r \text{ in all } \mu\vdash m}$

$\Rightarrow \sum_{\substack{r,s\geq 1\\ rs\leq m}} s\,m(r,s) = \sum_{\mu\vdash m}\sum_i k_i(\mu).$

On the other hand, the left-hand side equals:

$$\sum_{r,s} s\,m(r,s)=\sum_r\Big(\sum_s s\big(p(m-rs)-p(m-r(s+1))\big)\Big)=\sum_r\Big(\sum_s (s-(s-1))p(m-rs)\Big)=\sum_{r,s}p(m-rs)$$

$\Rightarrow$ eventually implies $\boxed{\sum_{\mu\vdash m}\sum_i k_i(\mu)=\sum_{\substack{r,s\geq 1\\ rs\leq m}} p(m-rs)}$ ∎

Exercise: Prove the formula for the coeff. K_m.

Theorem 2 (Kac): For $r,s\geq 1$, define $\boxed{h_{r,s}(c)=\frac{1}{48}\Big((13-c)(r^2+s^2)+\sqrt{(c-1)(c-25)}\,(r^2-s^2)-24rs-2+2c\Big)}$

Then $\det_{rs}(c,h_{r,s}(c))=0$ (more precisely, there is a degree rs singular vector in $M_{c,h_{r,s}(c)}$).

Rmk 3: (a) We shall prove this theorem later after the study of affine Lie alg-s (need $\widehat{\mathfrak{sl}}_2$-theory)

(b) By Lemma 1(b), we see that $\det_m(c,h_{r,s}(c))=0$ $\forall m\geq rs$.

(c) The above is true for any branch of $\sqrt{\cdots}$.

Now we are ready to prove the main result for today:

Theorem 3 (Kac, Feigin–Fuchs): For any $m\geq 1$, $\det_m(c,h)$ is explicitly given by:

$$\boxed{\det_m(c,h)=K_m\cdot\prod_{\substack{r,s\geq 1\\ rs\leq m}}\big(h-h_{r,s}(c)\big)^{p(m-rs)}}$$

$\leftarrow$ often called "Kac determinant formula".

Rmk 4: (a) Choosing branch of $\sqrt{\cdots}$ we should make the same choice for all $h_{\cdot,\cdot}$, in particular, $h_{r,s}(c)$ & $h_{s,r}(c)$.

(b) As $p(m-rs)=p(m-sr)$, the expression $(h-h_{r,s}(c))^{p(m-rs)}(h-h_{s,r}(c))^{p(m-sr)}$ does not depend on that choice $\sqrt{\cdots}$.

(c) For the above reasons, we see that $\det_m(c,h)$ is indeed polynomial in c,h.

▶ The proof is based on the following simple result:

Lemma 3: Let $A(t)$ be a matrix whose entries are polynomial in t and such that $\dim(\mathrm{Ker}(A(0)))\geq n$. Then $\det A(t)$ is divisible by t^n.

▶ Pick a basis $v_1,\ldots,v_n$ of $\mathrm{Ker}(A(0))$ and complete it to the basis of the entire vector space. Writing $A(t)$ in this basis, we see that the first n columns are divisible by t.

Hence, $\det A(t) \vdots t^n$. ∎

Using Lemma 3, we see that Theorem 2 $\Rightarrow$ Theorem 3. Indeed, Theorem 2 guarantees that if $h=h_{r,s}(c)$, then $M_{c,h_{r,s}(c)}$ has a singular vector w of degree $d=rs$. But, due to Problem 3 of Homework 2, the submodule of $M_{c,h_{r,s}(c)}$ generated by w is isom. to a Verma module $M_{c,h'}$. Moreover, it is clear that $w\in\mathrm{Ker}(\cdot,\cdot)$ implies $w'\in\mathrm{Ker}(\cdot,\cdot)$ $\forall w'\in U(Vir)w=M_{c,h'}$

(Continuation of the proof of Theorem 2)

Thus, the discussion from the previous paragraph (together with dimension f-la of graded components of $M_{c,h}$) implies that $\det_m(c,h)$ is divisible by $(h-h_{r,s}(c))^{p(m-rs)}$ $\forall r,s\geq 1$ s.t. $rs\leq m$.

But: For generic c, $\{h_{r,s}(c)\}_{r,s\geq 1}^{rs\leq m}$ are pairwise distinct

$$\Rightarrow \boxed{\det_m(c,h) \text{ is divisible by } \prod_{\substack{r,s\geq 1\\ rs\leq m}} (h-h_{r,s}(c))^{p(m-rs)}}$$

However, by Theorem 1, the leading term of $\det_m(c,h)$ is $K_m\cdot h^{\sum_{r,s\geq 1}^{rs\leq m} p(m-rs)}$

$$\Rightarrow \boxed{\det_m(c,h) = K_m\cdot \prod_{\substack{r,s\geq 1\\ rs\leq m}} (h-h_{r,s}(c))^{p(m-rs)}}$$

∎

Corollary 2: The Virasoro Verma module $M_{c,h}$ is irreducible iff (c,h) does not belong to:

* lines $\quad h-h_{r,r}(c)=0 \Leftrightarrow h+\frac{(r^2-1)(c-1)}{24}=0$
* quadrics $\quad (h-h_{r,s}(c))(h-h_{s,r}(c))=0$

$$\Updownarrow$$

$$\left(h-\frac{(r-s)^2}{4}\right)^2+\frac{h}{24}(c-1)(r^2+s^2-2)+\frac{1}{576}(r^2-1)(s^2-1)(c-1)^2+\frac{1}{48}(c-1)(r-s)^2(rs+1)=0.$$

Corollary 3: If $h>0$, $c>1$, then $M_{c,h}\simeq L_{c,h}$, i.e. $M_{c,h}$ is irreducible.

▸ It is clear from the above f-las of lines and quadrics that they do not contain points in the region $(h>0, c>1)$ as each summand is positive over there. ∎

Corollary 4: If $h\geq 0$, $c\geq 1$, then $L_{c,h}$ is unitary.

This was stated without proof as Theorem 1 in Lecture 5.

▸
- According to [Lecture 5, Corollary 5], we know that $L_{c,h}$ is unitary for $c\geq 1$, $h\geq \frac{1}{24}$.
- But due to Corollary 3, we have $L_{c,h}\simeq M_{c,h}$ for $c>1$, $h\geq 0$.

However, unitarity of $M_{c,h}$ is equivalent to the positivity of each matrix used to compute $\det_m(c,h)$.
Given a continuous family $\{A(t)\}$ of nondegenerate Hermitian matrices, s.t. $A(0)$ is positive definite, it is obvious that each $A(t)$ is also positive definite (b/c signature defines a continuous map to $\mathbb{Z}$ $\Rightarrow$ constant)
- Combining the above observations, we immediately see that $L_{c,h}\simeq M_{c,h}$ is unitary for $h>0, c>1$.

Finally, when we get to the boundary: $(h=0, c\geq 1)\cup(h\geq 0, c=1)$, then the same arguments imply that the Hermitian form $\langle\cdot,\cdot\rangle$ on $M_{c,h}$ is non-negative $\Rightarrow$ the induced Hermitian form $\langle\cdot,\cdot\rangle$ on $M_{c,h}/\mathrm{Ker}\langle\cdot,\cdot\rangle$ is positive $\Rightarrow$ unitary. But the latter quotient is exactly $L_{c,h}$ ∎

By a detailed analysis of the curves from Corollary 2, Friedan–Qiu–Shenker ('85, '86) proved that the only possible places of unitarity of $L_{c,h}$ (in the region $0\leq c<1$, $h\geq 0$) are the discrete set of points:

$$\boxed{\underbrace{\{(c(m), h_{r,s}(m)) \mid 1\leq s\leq r\leq m+1\}}_{\text{discrete series}}, \text{ where } c(m)=1-\frac{6}{(m+2)(m+3)} \Rightarrow h_{r,s}(m)=\frac{((m+3)r-(m+2)s)^2-1}{4(m+2)(m+3)}}$$

Rmk 5: (a) We will not prove that these are the only possible pts

(b) But we will establish unitarity condition for them later on.

Proposition 1: (a) If $c=0$, then $L_{c,h}$ is unitary iff $h=0$

(b) $L_{0,h} = M_{0,h}$ iff $h \neq \frac{m^2-1}{24}$ for all $m \in \mathbb{Z}_{\geq 0}$

(c) $L_{1,h} = M_{1,h}$ iff $h \neq \frac{m^2}{4}$ for all $m \in \mathbb{Z}_{\geq 0}$.

▶ (a) Let us compute $\det \begin{pmatrix} (L_{-N}^2 v_\lambda, L_{-N}^2 v_\lambda) & (L_{-N}^2 v_\lambda, L_{-2N} v_\lambda) \\ (L_{-2N} v_\lambda, L_{-N}^2 v_\lambda) & (L_{-2N} v_\lambda, L_{-2N} v_\lambda) \end{pmatrix}$ which should be in $\mathbb{R}_{\geq 0}$ given $L_{c,h}$-unitary

note that if $\exists (a_N, b_N) \in \mathbb{C}^2 \setminus (0,0)$ s.t. $(a_N L_{-2N} + b_N L_{-N}^2) v_\lambda$ belongs to J_λ, then we must have this $\det = 0$. However, as we'll see this may happen at most for one N if $h \neq 0$

But: $(L_{-2N} v_\lambda, L_{-2N} v_\lambda) = (v_\lambda, L_{2N} L_{-2N} v_\lambda) = 4N \cdot h + \frac{8N^3 - 2N}{12} \cdot c = 4Nh$ as $c=0$

$$(L_{-2N} v_\lambda, L_{-N}^2 v_\lambda) = (v_\lambda, L_{2N} L_{-N}^2 v_\lambda) = \underbrace{(v_\lambda, L_{-N} L_{2N} L_{-N} v_\lambda)}_{=0} + (v_\lambda, 3N \cdot L_N L_{-N} v_\lambda)$$

$$= 3N\left(2N \cdot h + \frac{N^3 - N}{12} \cdot c\right) = 6N^2 h \quad \text{as } c=0$$

$$(L_{-N}^2 v_\lambda, L_{-N}^2 v_\lambda) = (v_\lambda, L_N^2 L_{-N}^2 v_\lambda) = (v_\lambda, L_N L_{-N} L_N L_{-N} v_\lambda) + \left(v_\lambda, L_N \left(2N \cdot L_0 + \frac{N^3 - N}{12} c\right) L_{-N} v_\lambda\right)$$

$$= 2\left(2Nh + \frac{N^3 - N}{12} c\right)^2 + 2N \cdot N \cdot \left(2Nh + \frac{N^3 - N}{12} c\right) = 8N^2 h^2 + 4N^3 h \quad \text{as } c=0$$

So: Above $\det = (8N^2h^2 + 4N^3 h) \cdot 4Nh - (6N^2 h)^2 = 32 N^3 h^3 - 20 N^4 h^2 = \underbrace{4N^3 h^2 (8h - 5N)}$,

↑ it becomes <0 for $N \gg 1$ unless $h=0$.

On the other hand $L_{0,0}$ – trivial $\Rightarrow$ unitary!

(c) Looking at Corollary 2, we see that $L_{1,h} = M_{1,h}$ iff $(1,h)$ is not on the listed:

$\left.\begin{array}{l} * \text{ lines} \Leftrightarrow h \neq 0 \\ * \text{ quadrics} \Leftrightarrow h - \frac{(r-s)^2}{4} \neq 0 \ (r \neq s) \end{array}\right\}$ So: $M_{1,h}$ – irred. $\Leftrightarrow h \neq \frac{m^2}{4}$ $\forall m \in \mathbb{Z}$.

(b) Explicit f-la for $h_{r,s}(c)$ yields (upon usual branch of $\sqrt{\cdots}$)

$$h_{r,s}(0) = \frac{1}{48}\left(13(r^2+s^2) + 5(r^2-s^2) - 24rs - 2\right) = \frac{18r^2 + 8s^2 - 24rs - 2}{48} = \frac{(3r-2s)^2 - 1}{24}$$

Since $3r - 2s$ can be any integer as $r, s \geq 1$, we get:

$M_{0,h}$ – irreducible iff $h \neq \frac{m^2-1}{24}$ $\forall m \in \mathbb{Z}$ ∎

Rmk 5: (a) If $m=0$, then $c(0)=0$ and $h_{1,1}(0)=0$, which agrees with Prop 1(a)

(b) If $m=1$, then $c(1) = \frac{1}{2}$ and $h_{1,1}(1) = 0$, $h_{2,1}(1) = \frac{1}{2}$, $h_{2,2}(1) = \frac{1}{16}$ $\Rightarrow$ Friedan-Qiu-Shenker guarantee that $L_{1/2,h}$ may be unitary only for $h = 0, \frac{1}{16}, \frac{1}{2}$. But in Lecture 5, we saw that these modules are unitary indeed.

Rmk 6: A quite similar determinant formula for the case of f.dim. semisimple Lie algebras $\mathfrak{g}$ was first established by Shapovalov ('72) and its proof was improved by Jantzen ('77).

$$\boxed{\text{"} \det_\eta = \prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} \left(h_\alpha + \rho(h_\alpha) - n \frac{(\alpha,\alpha)}{2}\right)^{P(\eta - n\alpha)} \text{"}}$$

$P(\eta)$ – Kostant partition function
$\det_\eta$: we are looking at degree $-\eta \in Q$ component of $U(\mathfrak{n}_-)$

A generalization of this formula for arbitrary contragredient Lie alg-s was established by Kac-Kazhdan ('79). We shall get back to that latter in the class.