— LECTURE 12 —

Today: From $\bar{\mathfrak{a}}_\infty$ to affine Lie algebras
(generalizing the construction of the Heisenberg algebra $\mathcal{A} \hookrightarrow \mathfrak{a}_\infty$)

- Consider the loop algebra $L\mathfrak{gl}_n = \mathfrak{gl}_n[t,t^{-1}] = \mathfrak{gl}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$. Any element of $L\mathfrak{gl}_n$ may be written as
$$\boxed{a(t) = \sum_{k\in\mathbb{Z}} a_k t^k,\ a_k \in \mathfrak{gl}_n \text{ and only finitely many of them are nonzero}}$$

Note that it has a natural basis $\{E_{ij}(k) := t^k \cdot E_{ij} \mid 1 \le i,j \le n,\ k \in \mathbb{Z}\}$, in which the Lie bracket is:
$$\boxed{[E_{ij}(k), E_{i'j'}(k')] = \delta_{i'j} \cdot E_{ij'}(k+k') - \delta_{ij'} \cdot E_{i'j}(k+k')}$$

Let $V_0 = \mathbb{C}^n$ be the natural 1st fundamental repr-n of $\mathfrak{gl}_n$ (realized via columns of height n) with a natural basis $\{e_1, e_2, \ldots, e_n\}$. Then, we get
$$\boxed{L\mathfrak{gl}_n \curvearrowright \mathbb{C}^n[t,t^{-1}]} \longleftarrow \text{with a basis } \{e_i \cdot t^k \mid 1 \le i \le n,\ k \in \mathbb{Z}\}$$

- Denote $e_i \cdot t^k \rightsquigarrow v_{i-kn}$, so that $\mathbb{C}^n[t,t^{-1}]$ gets identified with $V = \mathbb{C}^\infty$ – tautological $\mathfrak{gl}_\infty$-repr-n.

<u>Note</u>: $E_{ij}(k)\, v_{j'+nk'} = \delta_{jj'} \cdot v_{i+n(k'-k)}$

Therefore, having identified $\mathbb{C}^n[t,t^{-1}]$ with $\mathbb{C}^\infty$, we obtain a natural embedding
$$\boxed{L\mathfrak{gl}_n \overset{\tau}{\hookrightarrow} \bar{\mathfrak{a}}_\infty \text{ given explicitly by } \sum_k a_k t^k \overset{\tau}{\longmapsto} \text{Block Matrix} \begin{pmatrix} \ddots & a_1 & a_2 & & \ddots \\ \ddots & a_0 & a_1 & a_2 & \\ \ddots & a_{-1} & a_0 & a_1 & a_2 \\ & a_{-2} & a_{-1} & a_0 & \ddots \\ & & a_{-2} & a_{-1} & \ddots \end{pmatrix}}$$

(From the above formula, we see that $\tau(E_{ij}(k)) = \sum_{m\in\mathbb{Z}} E_{nm+i,\, n(m+k)+j}$)

Remark 1: (a) τ is compatible with multiplication (where we view both $L\mathfrak{gl}_n$, $\bar{\mathfrak{a}}_\infty$ as assoc. alg-s)

(b) Let ω be an anti-linear anti-involution on $L\mathfrak{gl}_n$ sending $X \cdot t^k \longmapsto X^+ \cdot t^{-k}$, where X^+ denotes Hermitian adjoint of $X \in \mathfrak{gl}_n$, see Lecture 4. Then:
$$\boxed{\tau(\omega(a(t))) = (\tau(a(t)))^\dagger} \longleftarrow \text{matrix Hermitian adjoint in } \bar{\mathfrak{a}}_\infty.$$

(c) $\tau\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ t & & & 0 \end{pmatrix} = T \overset{(a)}{\Longrightarrow} \boxed{\tau\left(\begin{pmatrix} & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ t & & & \end{pmatrix}^k\right) = T^k \quad \forall k \in \mathbb{Z}}$

Recalling that $\bar{\mathfrak{a}}_\infty$ has a natural 1-dim central extension $\mathfrak{a}_\infty$ via the 2-cocycle $\alpha: \bar{\mathfrak{a}}_\infty \times \bar{\mathfrak{a}}_\infty \to \mathbb{C}$ (see Lecture 7), we naturally obtain an embedding of 1-dim central extension of $L\mathfrak{gl}_n$ (via the restriction of α_τ) into $\mathfrak{a}_\infty$.

<u>Lemma 1</u>: This is exactly the 1-dim central extension of $L\mathfrak{gl}_n$ defining affine Lie algebras, i.e.
$$\alpha_\tau(a(t), b(t)) = \underset{[t^{-1}]\text{-coeff}}{\underline{\mathrm{Res}_{t=0}}}(a'(t), b(t))\,dt = \sum_k k \cdot \mathrm{Tr}(a_k b_{-k})$$

► $\alpha_\tau(E_{ij}(k), E_{i'j'}(k')) = \alpha(\tau(E_{ij}(k)), \tau(E_{i'j'}(k'))) = k \cdot \delta_{ij'} \delta_{i'j} \delta_{k+k',0}$

$\Rightarrow \alpha_\tau(a \cdot t^k, b \cdot t^{k'}) = \delta_{k+k',0} \cdot k \cdot \mathrm{Tr}(ab)$. This completes the proof. ∎

So: $\boxed{\widehat{gl}_n := Lgl_n \oplus \mathbb{C}\cdot K \hookrightarrow \mathfrak{a}_\infty}$

Likewise, starting from $Lsl_n = sl_n[t,t^{-1}] \hookrightarrow \bar{\mathfrak{a}}_\infty$, we get $\boxed{\widehat{sl}_n := Lsl_n \oplus \mathbb{C}\cdot K \hookrightarrow \mathfrak{a}_\infty}$

Recalling natural action $\mathfrak{a}_\infty \curvearrowright \mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)}$, we obtain:

Corollary 1: The spaces $\mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)}$ become $\widehat{gl}_n$- and $\widehat{sl}_n$-modules of level 1
i.e. K acts via $1\cdot \mathrm{Id}$.

- Recall that $(X|Y) = \mathrm{Tr}(XY)$ is a symmetric nondeg. invariant bilinear form on gl_n.
Likewise, $(a(t)|b(t)) = \mathrm{Res}_{t=0} \mathrm{Tr}(a(t)b(t))\frac{dt}{t}$ is a symm. nondeg. invariant bilinear form on Lgl_n.
We extend this pairing to $(\cdot|\cdot)$ on $\widehat{gl}_n$ by defining $(K|Lgl_n) = 0 = (K|K)$.
$\overset{\shortparallel}{(Lgl_n|K)}$
Then it is symm. inv. bilinear form on $\widehat{gl}_n$, but degenerate!

Let us now slightly enlarge $\widehat{gl}_n$. Let us consider the derivation of $\widehat{\mathfrak{g}}$ ($\mathfrak{g} = gl_n$ or sl_n so far):
$\boxed{d: \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}} \text{ given by } K \mapsto 0,\ a(t) \mapsto ta'(t) \text{ i.e. } X\cdot t^k \mapsto k\cdot Xt^k.}$

Then we can form a semidirect product $\tilde{\mathfrak{g}} := \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$. ← also called affine Lie algebra

Note: Abstractly, we just may view $\widetilde{gl}_n = \widehat{gl}_n \oplus \mathbb{C}d$ with $[d,K]=0$, $[d, X\cdot t^k] = k\cdot Xt^k$ ($X \in gl_n$)

Lemma 2: Extending the above pairing on $\widehat{gl}_n$ to that on $\widetilde{gl}_n$ via
$$(d|d)=0,\ (d|K)=(K|d)=1,\ (d|a(t))=(a(t)|d)=0$$
gives rise to a symm. invariant nondegenerate pairing on $\widetilde{gl}_n$.

► The nondegeneracy is clear as $(d|K)=1$. To prove invariance, it suffices to verify
(1) $([d, X\cdot t^k] | y\cdot t^\ell) = -(Xt^k | [d, yt^\ell])$ ← follows from the fact that both sides are zero unless $k+\ell=0$
(2) $([X\cdot t^k, d] | y\cdot t^\ell) = -(d | [Xt^k, yt^\ell])$ ← follows from the 2-cocycle f-la ■

- Let us now recall that $\widehat{sl}_n, \widehat{gl}_n$, hence, $\widetilde{sl}_n, \widetilde{gl}_n$ are $\mathbb{Z}$-graded Lie alg-s via the "principal gradation".
In particular, we have the following triangular decomposition:

$$\boxed{\begin{aligned} \widetilde{sl}_n = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_-, \text{ where } \ & \widetilde{\mathfrak{n}}_+ = \mathfrak{n}_+ + \textstyle\sum_{k>0} t^k\cdot sl_n, \quad \mathfrak{n}_+ \subset sl_n - \text{strictly upper-}\Delta \\ & \widetilde{\mathfrak{n}}_- = \mathfrak{n}_- + \textstyle\sum_{k>0} t^{-k}\cdot sl_n, \quad \mathfrak{n}_- \subset sl_n - \text{strictly lower-}\Delta \\ & \widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}\cdot K \oplus \mathbb{C}\cdot d, \quad \mathfrak{h} \subset sl_n - \text{diagonal} \end{aligned}}$$

$\mathfrak{h}$ has a basis $\{h_i := E_{ii} - E_{i+1,i+1} \mid 1 \le i \le n-1\} \Rightarrow \{h_i,\ h_0 := K - (\underbrace{h_1 + \ldots + h_{n-1}}_{E_{11}-E_{nn}}), d\}_{i=1}^{n-1}$ - basis of $\widetilde{\mathfrak{h}}$.

Def 1: The elements $\{\tilde{\omega}_i\}_{i=0}^{n-1} \subset \widetilde{\mathfrak{h}}^*$ are defined via $\tilde{\omega}_i(h_j) = \delta_{ij}$ ($0 \le j \le n-1$), $\tilde{\omega}_i(d) = 0$.
Likewise, we also define weights $\{\tilde{\omega}_m\}_{m\in\mathbb{Z}}$ of $\widetilde{gl}_n$:

Def 2: The elements $\{\tilde{\omega}_m\}_{m\in\mathbb{Z}} \subset (\widetilde{\mathfrak{h}}_{gl_n})^* = (\mathrm{span}\langle E_{11}, \ldots, E_{nn}, K, d\rangle)^*$ are defined via
$\boxed{\tilde{\omega}_m(d) = 0,\ \tilde{\omega}_m(K) = 1,\ \tilde{\omega}_m(E_{ii}) = \begin{cases} 1, & \text{if } i \le \bar{m} \\ 0, & \text{if } i > \bar{m} \end{cases} + \frac{m - \bar{m}}{n}}$, where $\bar{m} = m \bmod n \in \{0,1,\ldots,n-1\}$

Remark 2: Note that $\forall m \in \mathbb{Z}$, the restriction of $\tilde{\omega}_m \in (\widetilde{\mathfrak{h}}_{gl_n})^*$ to $\widetilde{\mathfrak{h}}_{sl_n} \subseteq \widetilde{\mathfrak{h}}_{gl_n}$ equals $\tilde{\omega}_{\bar{m}}$.

• To proceed further, we need to extend $\widehat{\mathfrak{gl}}_n$-action on $\mathcal{F}^{(m)}$ to that of $\widetilde{\mathfrak{gl}}_n$.

<u>Lemma 3</u>: There exists a unique extension of $\widehat{\mathfrak{gl}}_n$-representation $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}}_n$ such that $d(\psi_m)=0$. It is explicitly given in the basis of semiinfinite elementary wedges by

$$\boxed{d(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots)=\Big(\sum_{k\geq 0}\Big(\Big\lceil\frac{m-k}{n}\Big\rceil-\Big\lceil\frac{i_k}{n}\Big\rceil\Big)\Big)\cdot v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots}$$

<u>Exercise 1</u>: Prove this Lemma.

<u>Proposition 1</u>: For $m\in\mathbb{Z}$, the $\widetilde{\mathfrak{gl}}_n$-module $\mathcal{F}^{(m)}$ (of Lemma 3) is irreducible with highest weight $\tilde{\omega}_m$.

▸ • As we pointed out before the image of $L\mathfrak{gl}_n$ in $\bar{a}_\infty$ contains all $\{T^i \mid i\in\mathbb{Z}\}$, hence, $\widetilde{\mathfrak{gl}}_n\supset\widehat{\mathfrak{gl}}_n$ contain the Heisenberg subalg. $\mathcal{A}$. But as we saw before $\mathcal{F}^{(m)}\simeq\mathcal{B}^{(m)}$ is already an irreducible repr. w.r.t. this $\mathcal{A}$-action. Hence, $\mathcal{F}^{(m)}$ is an irreducible $\widetilde{\mathfrak{gl}}_n$-representation.

• As $\tilde{n}_+=n\oplus\sum_{k>0}t^k\cdot\mathfrak{gl}_n$ obviously embeds into a_∞ as strictly upper-Δ matrices, we immediately get $\tilde{\rho}(\tilde{n}_+)\psi_m=0$.

• It remains to show $\tilde{\rho}(h)\psi_m=\tilde{\omega}_m(h)$ $\forall h\in\tilde{\mathfrak{h}}_{\mathfrak{gl}_n}$. For $h=K$ or d, this is clear. Hence, it suffices to verify $\tilde{\rho}(E_{ii})\psi_m=\left(\begin{cases}1, & \text{if } i\leq\bar{m}\\ 0, & \text{if } i>\bar{m}\end{cases}+\frac{m-\bar{m}}{n}\right)\cdot\psi_m$

But $\tau(E_{ii})=\sum_{j\equiv i \bmod n}E_{jj}$ and $\tilde{\rho}(E_{jj})\psi_m=(\delta_{j\leq m}-\delta_{j\leq 0})\cdot\psi_m$.

<u>So</u>: ◦ if $m>0$ $\Rightarrow$ $\tilde{\rho}(E_{ii})\psi_m=\#\{0<j\leq m \mid j\equiv i \bmod n\}\cdot\psi_m=\tilde{\omega}_m(E_{ii})\cdot\psi_m$

◦ if $m\leq 0$ $\Rightarrow$ $\tilde{\rho}(E_{ii})\psi_m=-\#\{m<j\leq 0 \mid j\equiv i \bmod n\}\cdot\psi_m=\tilde{\omega}_m(E_{ii})\cdot\psi_m$

• Finally, ψ_m generates the entire $\mathcal{F}^{(m)}$ over $\mathcal{A}$-action $\Rightarrow$ also over $\widetilde{\mathfrak{gl}}_n$-action.

This completes the proof of Proposition. ∎

<u>Note</u> that $\mathcal{F}^{(m)}\simeq\mathcal{B}^{(m)}$ via boson-fermion correspondence $\Rightarrow$ $\mathcal{B}^{(m)}$ is also an irred. $\widetilde{\mathfrak{gl}}_n$-module.

• Let us now recall $\mathfrak{sl}_n\subset\mathfrak{gl}_n$, $L\mathfrak{sl}_n\subset L\mathfrak{gl}_n$, $\widehat{\mathfrak{sl}}_n\subset\widehat{\mathfrak{gl}}_n$, $\widetilde{\mathfrak{sl}}_n\subset\widetilde{\mathfrak{gl}}_n$

Q: What can we say about the $\widetilde{\mathfrak{sl}}_n$-action on $\mathcal{F}^{(m)}$?

First note that $\{T^{ni}\}_{i\in\mathbb{Z}}$ commute with the action of $\widetilde{\mathfrak{sl}}_n$ (as T^{ni} corresponds to $I_{n\times n}\cdot t^i$ which clearly commutes with $\widetilde{\mathfrak{sl}}_n$ inside $\widetilde{\mathfrak{gl}}_n$), but their action on $\mathcal{F}^{(m)}$ is not given by scalars $\Rightarrow$ $\mathcal{F}^{(m)}$ is <u>not</u> an irreducible $\widetilde{\mathfrak{sl}}_n$-module.

Consider a Lie subalgebra $\mathcal{A}^{(n)}\subset\mathcal{A}$ generated (=spanned) by $\{a_{ni}\}_{i\in\mathbb{Z}}\cup\{K\}$

<u>Lemma 4</u>: The assignment $K\longmapsto nK$, $a_i\longmapsto a_{ni}$ gives rise to Lie alg. isomorphism $\mathcal{A}\xrightarrow{\sim}\mathcal{A}^{(n)}$.

The following simple result establishes an explicit relation b/w $\widetilde{\mathfrak{gl}}_n$ and $\widetilde{\mathfrak{sl}}_n$ together with $\mathcal{A}^{(n)}$.

<u>Lemma 5</u>: The Lie algebras $\widetilde{\mathfrak{gl}}_n$ and $(\widetilde{\mathfrak{sl}}_n\oplus\mathcal{A}^{(n)})/(K_1-K_2)$ are isomorphic

(here $K_1=(K,0)$ and $K_2=(0,K)$, while $K_1-K_2=0$ means that the central el's act in the same way)

<u>Exercise 2</u>: Prove this Lemma.

(the map $\widetilde{\mathfrak{sl}}_n\oplus\mathcal{A}^{(n)}/(K_1-K_2)\longrightarrow\widetilde{\mathfrak{gl}}_n$ maps $(Xt^k,0)\longmapsto Xt^k$ $(X\in\mathfrak{sl}_n, k\in\mathbb{Z})$, $a_{ni}\longmapsto I_{n\times n}\cdot t^i$, $K_1=K_2\longmapsto K$)

This result allows to view the $\mathfrak{gl}_n$-module $\mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)}$ as an $\widehat{\mathfrak{sl}}_n \oplus \mathcal{A}^{(n)}$-module where K_1 & K_2 act in the same way. Recall that the generators of $\mathcal{A}^{(n)}$ act via $\{T^{ni}\}_{i\in\mathbb{Z}}$ (which commute with $\widehat{\mathfrak{sl}}_n$). Define

$$\boxed{\mathcal{B}_0^{(m)} := \{x \in \mathcal{B}^{(m)} \mid T^{ni}(x)=0 \ \forall i>0\} \xlongequal[\text{via } k\partial_{x_k}]{\text{as } T^k \text{ acts}} \mathbb{C}[\{x_j\}_{j\in\mathbb{Z}_{>0},\, j\nmid n}].}$$

Proposition 2: For $m\in\mathbb{Z}$, $\mathcal{B}_0^{(m)}$ is an irreducible $\widehat{\mathfrak{sl}}_n$-module (also $\widehat{\mathfrak{sl}}_n$-module) with the highest weight $\tilde{\omega}_{\bar{m}}$ (here $\bar{m}\in\{0,1,\ldots,n-1\}$ is the remainder of m mod n). Moreover, $\mathcal{B}^{(m)} \simeq \mathcal{B}_0^{(m)} \otimes \tilde{F}_m$ as $\widehat{\mathfrak{sl}}_n \oplus \mathcal{A}^{(n)}$-modules, where $\tilde{F}_m$ - appropriate Fock module/$\mathcal{A}^{(n)}$.

► Set $\tilde{F}_m = \mathbb{C}[x_n, x_{2n}, \ldots]$, so that we have a tautological isom. of vector spaces $\mathcal{B}^{(m)} = \mathbb{C}[\{x_j\}_{j\geq 1}] \simeq \mathbb{C}[\{x_j\}_{j\geq1,\, n\nmid j}] \otimes \mathbb{C}[\{x_j\}_{n|j}] = \mathcal{B}_0^{(m)} \otimes \tilde{F}_m$. Note that $\mathcal{A}^{(n)}$ acts on the second tensor factor, while $\widehat{\mathfrak{sl}}_n$ acts on the first tensor factor.

As the resulting $\widehat{\mathfrak{gl}}_n \simeq \widehat{\mathfrak{sl}}_n \oplus \mathcal{A}^{(n)}/(K_1-K_2)$-module $\mathcal{B}^{(m)}$ is irreducible $\Rightarrow \mathcal{B}_0^{(m)}$ - irred. $\widehat{\mathfrak{sl}}_n$-module (alternatively, we could appeal to the Heisenberg alg. gen-d by $\{T^j\}_{n\nmid j} \cup \{K\}$). ■

• Finally, we may derive a classification of unitary highest weight $\widehat{\mathfrak{sl}}_n$-representations.

Theorem 1: The highest weight $\widehat{\mathfrak{sl}}_n$-repr. L_λ is unitary iff $\lambda = k_0\tilde{\omega}_0 + \ldots + k_{n-1}\tilde{\omega}_{n-1}$ with $k_i\in\mathbb{Z}_{\geq 0}$.

► • First, $\forall m\in\{0,1,\ldots,n-1\}$ $L_{\tilde{\omega}_m}$ is unitary, since the contravariant form on $L_{\tilde{\omega}_m}$ arises from the contravariant form on $\mathcal{B}^{(m)}$ that is unitary.

• As $\{L_{\tilde{\omega}_m}\}_{0\leq m\leq n-1}$ - unitary $\xRightarrow{k_i\in\mathbb{Z}_{\geq0}}$ $L_{\tilde{\omega}_0}^{\otimes k_0} \otimes \ldots \otimes L_{\tilde{\omega}_{n-1}}^{\otimes k_{n-1}}$ - unitary $\Rightarrow L_{k_0\tilde{\omega}_0+\ldots+k_{n-1}\tilde{\omega}_{n-1}}$ - unitary as a composition factor (actually, as a summand).

• Finally, we want to show that unitarity of L_λ implies that $\lambda = k_0\tilde{\omega}_0+\ldots+k_{n-1}\tilde{\omega}_{n-1}$ with $k_i\in\mathbb{Z}_{\geq0}$.

Recall (see Lemma 1 of Lecture 7) that the irreducible $\mathfrak{sl}_2$-module L_μ is unitary iff $\mu\in\mathbb{Z}_{\geq0}$.
(w.r.t. antilinear anti-involution $t: \mathfrak{sl}_2 \to \mathfrak{sl}_2$ s.t. $e_i^t = f_i$, $f_i^t = e_i$, $h_i^t = h_i$)

But for every $i\in\{0,1,\ldots,n-1\}$, we have an $\mathfrak{sl}_2$-subalg. $\mathfrak{sl}_2^{(i)} \subseteq \widehat{\mathfrak{sl}}_n$ s.t. $t_{\widehat{\mathfrak{sl}}_n}|_{\mathfrak{sl}_2^{(i)}} = t_{\mathfrak{sl}_2}$.

Indeed,

$$\boxed{\begin{array}{ll} \text{for } 1\leq i\leq n-1: & \mathfrak{sl}_2^{(i)} = \langle e_i = E_{i,i+1},\ f_i = E_{i+1,i},\ h_i = E_{ii} - E_{i+1,i+1}\rangle \\ \text{for } i=0: & \mathfrak{sl}_2^{(0)} = \langle e_0 = E_{n,1}\cdot t,\ f_0 = E_{1n}\cdot t^{-1},\ h_0 = K + E_{nn} - E_{11}\rangle \end{array}}$$

Therefore, unitarity of L_λ implies $\lambda(h_i)\in\mathbb{Z}_{\geq0}$ $\forall i\in\{0,1,\ldots,n-1\}$.

But: $\lambda = \sum_{i=0}^{n-1}\lambda(h_i)\tilde{\omega}_i$ (as $\tilde{\omega}_i$ - dual to h_i) $\Rightarrow \lambda = \sum_{i=0}^{n-1} k_i\tilde{\omega}_i$ with $k_i\in\mathbb{Z}_{\geq0}$. ■

Remark 3: If $\lambda = k_0\tilde{\omega}_0+\ldots+k_{n-1}\tilde{\omega}_{n-1}$ ($k_i\in\mathbb{Z}_{\geq0}$) so that L_λ is a unitary $\widehat{\mathfrak{sl}}_n$-module, then $U(\mathfrak{sl}_n)v_\lambda =: L_{\bar{\lambda}}$ (with $\bar{\lambda} = k_1\omega_1 + k_2\omega_2 + \ldots + k_{n-1}\omega_{n-1}$) as $\mathfrak{sl}_n$-modules.

Def 3: The value $\lambda(K) = \lambda(h_0+h_1+\ldots+h_{n-1}) = k_0+k_1+\ldots+k_{n-1}$ is the *level* of $L(\lambda)$.