**Today**: Sugawara Construction
(generalizing our previous construction of $Vir \curvearrowright F_\mu$).

- Let $\mathfrak{g}$ – fin. dim. Lie algebra/$\mathbb{C}$ with an invariant symmetric bilinear form $(\cdot,\cdot)$ (not necessarily nondegenerate!)

$\rightsquigarrow L\mathfrak{g} = \mathfrak{g}[t,t^{-1}] \rightsquigarrow \boxed{\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}\cdot K}$

the 2-cocycle used there is from Lecture 1: $\boxed{\alpha(a(t), b(t)) = \mathrm{Res}_{t=0}(a'(t), b(t))dt}$

**Def 1**: $k \in \mathbb{C}$ is called non-critical for $(\mathfrak{g}, (\cdot,\cdot))$ iff $k\cdot(\cdot,\cdot) + \frac{1}{2}\underbrace{Kil}$ is nondegenerate

Killing form: $Kil(a,b) = \mathrm{Tr}_{\mathfrak{g}}(ad(a)\,ad(b))$

**Def 2**: A $\hat{\mathfrak{g}}$-module $M$ is called admissible if $\forall v \in M\ \exists N:\ at^n(v) = 0\ \forall a \in \mathfrak{g}, n \geq N$

The following result is analogous to [Lemma 1 of Lecture 1]:

**Lemma 1**: There is a natural homomorphism $\eta_{\hat{\mathfrak{g}}}: W \longrightarrow \mathrm{Der}\,\hat{\mathfrak{g}}$ defined via

$$\boxed{\eta_{\hat{\mathfrak{g}}}(f\partial_t)(g,\alpha) = (fg', 0)\ \forall\ f \in \mathbb{C}[t,t^{-1}], g \in \mathfrak{g}[t,t^{-1}], \alpha \in \mathbb{C}}$$

Exercise: Prove it!

As a result, we may form semidirect product $W \ltimes \hat{\mathfrak{g}}$ and $Vir \ltimes \hat{\mathfrak{g}}$ (here $Vir \twoheadrightarrow W \to \mathrm{Der}\,\hat{\mathfrak{g}}$).

**Theorem 1** (Sugawara Construction): Let $k \in \mathbb{C}$ be non-critical for $(\mathfrak{g}, (\cdot,\cdot))$ and $M$ be an admissible $\hat{\mathfrak{g}}$-module of level $k$. Then the action $\hat{\mathfrak{g}} \curvearrowright M$ extends to an action $Vir \ltimes \hat{\mathfrak{g}} \curvearrowright M$ with the generators $L_n$ of $Vir$ acting via

$$\boxed{L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{a\in B} :a_m a_{n-m}:}$$

with the central charge

$$\boxed{c = k\cdot\sum_{a\in B}(a,a)}.$$

Here $B$ is a basis of $\mathfrak{g}$ orthonormal w.r.t. $k(\cdot,\cdot) + \frac{1}{2}Kil$, $a_n := at^n \in \hat{\mathfrak{g}}$ for $a \in \mathfrak{g}$, and the normal ordered product $:a_m a_\ell:$ is defined via $:a_m a_\ell: = \begin{cases} a_m a_\ell, & \text{if } m \leq \ell \\ a_\ell a_m, & \text{if } m > \ell \end{cases}$.

**Remark 1**:

(a) First, we note that $L_n(v)$ is well-defined $\forall n \in \mathbb{Z}$ $\forall v \in M$ as $M$ is admissible.

(b) We also note that $L_n$ does not depend on the choice of $B$.

(c) Finally, let us observe that in the case $\mathfrak{g} = \mathbb{C}$, $(x,y) = xy$, $k = 1$, $M = F_\mu$ (as $\hat{\mathfrak{g}} \simeq \mathcal{A}$), we precisely recover our previous construction $Vir \ltimes \mathcal{A} \curvearrowright F_\mu$ from Lecture 5.

To prove Theorem 1, we need to verify:

$$\boxed{\begin{aligned} &(1)\ [L_n, b_r] = -r\,b_{n+r} \\ &(2)\ [L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\cdot k\sum_{a\in B}(a,a) \end{aligned}}$$

► (Proof of Theorem 1)

We will need the following two simple Lemmas.

- **Lemma 2**: For any $x\in\mathfrak{g}$, we have $\boxed{\sum_{a\in B}[x,a]a+a[x,a]=0}$.

**Lemma 2'**: $\sum_{a\in B}[x,a]\otimes a+a\otimes[x,a]=0$

► Proof is exactly as of Lemma 2 ∎

**Remark 2**: The more invariant way to state this is by saying $\boxed{[\mathfrak{g},\Omega_0]=0}$ with $\Omega_0=\sum_i a_i a^i$ - Casimir elt ($\{a_i\}$, $\{a^i\}$ - dual bases of $\mathfrak{g}$).

► (Proof of Lemma 2)

Let $(\cdot|\cdot)$ be the nondeg. form $k(\cdot,\cdot)+\frac{1}{2}Kil$, w.r.t. which $B$ - orthonormal. Then:

$$\sum_{a\in B}a[x,a]=\sum_{a,a'\in B}a\cdot([x,a]|a')\cdot a'=-\sum_{a,a'\in B}a\cdot(a|[x,a'])a'=-\sum_{a'\in B}[x,a']a'=-\sum_{a\in B}[x,a]a$$ ∎

- **Lemma 3**: For $x\in\mathfrak{g}$, we have $\boxed{\sum_{a\in B}[a,[a,x]]=\sum_{a\in B}Kil(x,a)a}$

► Let us recall how $Kil(\cdot,\cdot)$ is computed. Pick a basis $\{c_1,\ldots,c_m\}$ of $\mathfrak{g}$ and let $\{c_1^*,\ldots,c_m^*\}$ be the dual basis of $\mathfrak{g}^*$, so that $Kil(x,a)=\sum_{j=1}^m c_j^*([x,[a,c_j]])$. Thus:

$$\sum_{a\in B}Kil(x,a)\cdot a=\sum_{a\in B}\sum_{j=1}^m c_j^*([x,[a,c_j]])\cdot a=\sum_{j=1}^m\Big(\sum_{a\in B}c_j^*([x,[a,c_j]])\cdot a\Big)\ominus$$

**But**: $\sum_{a\in B}[a,c_j]\otimes a\overset{\text{Lemma 2'}}{=\!=}-\sum_{a\in B}a\otimes[a,c_j]$

$$\ominus-\sum_{j=1}^m\sum_{a\in B}c_j^*([x,a])\cdot[a,c_j]=-\sum_{a\in B}[a,\sum_{j=1}^m c_j^*([x,a])\cdot c_j]=-\sum_{a\in B}[a,[x,a]]=\sum_{a\in B}[a,[a,x]]$$ ∎

**Corollary 1**: For $x\in\mathfrak{g}$, we have $\boxed{x=k\sum_{a\in B}(x,a)a+\frac{1}{2}\sum_{a\in B}[[x,a],a]}$

► Combine the fact that $\{a\}_{a\in B}$ - orthonormal basis of $\mathfrak{g}$ w.r.t. $k(\cdot,\cdot)+\frac{1}{2}Kil$ with Lemma 3 ∎

- Now we are ready to prove

$$\boxed{[b_r,L_n]=rb_{n+r}\quad\forall r,n\in\mathbb{Z}\quad\forall b\in\mathfrak{g}}$$

First, we note $L_n=\frac{1}{2}\sum_{\substack{a\in B\\ m\in\mathbb{Z}}}:a_m a_{n-m}:=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{m:|m-\frac{n}{2}|\le N}:a_m a_{n-m}:$. Thus:

$$[b_r,L_n]=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}\big([b_r,a_m]a_{n-m}+a_m[b_r,a_{n-m}]\big)\ominus$$

**But**: $[x_i,y_j]=[x,y]_{i+j}+K\cdot\alpha(x_i,y_j)$ and $K$ acts on $M$ as $k\cdot Id_M$.

$$\ominus\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}\big([b,a]_{r+m}a_{n-m}+a_m\cdot[b,a]_{r+n-m}+K\cdot\alpha(b_r,a_m)\cdot a_{n-m}+K\cdot a_m\cdot\alpha(b_r,a_{n-m})\big)$$

$$=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}\big([b,a]_{r+m}\cdot a_{n-m}+a_m\cdot[b,a]_{r+n-m}\big)+\frac{1}{2}\sum_{a\in B}2k\cdot r\cdot(b,a)\cdot a_{n+r}$$

level = action of K

$$\overset{\text{Lemma 2}}{=\!=}\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}\big([b,a]_{r+m}\cdot a_{n-m}-[b,a]_m\cdot a_{r+n-m}\big)+rk\sum_{a\in B}(b,a)\cdot a_{n+r}$$

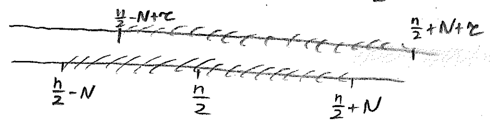
$$=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Big(\text{"}\sum_{|m-r-\frac{n}{2}|\le N}-\sum_{|m-\frac{n}{2}|\le N}\text{"}\Big)[b,a]_m\cdot a_{r+n-m}+rk\sum_{a\in B}(b,a)\cdot a_{n+r}\ominus$$

Note that "$\sum-\sum$" above simplifies. To write it carefully, assume $r\ge0$ (the case $r<0$ is analogous).

(Continuation of the proof of Theorem 1)

Assuming $r \geq 0$: " $\sum_{|m-r-\frac{n}{2}|\le N} - \sum_{|m-\frac{n}{2}|\le N}$ " = " $\sum_{\frac{n}{2}+N<m\le \frac{n}{2}+N+r} - \sum_{\frac{n}{2}-N\le m<\frac{n}{2}-N+r}$ " for $N \gg 0$,

which follows from the picture :




$$\ominus\ \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Big(-\sum_{\frac{n}{2}-N\le m<\frac{n}{2}-N+r}[b,a]_m\cdot a_{r+n-m}+\sum_{\frac{n}{2}+N<m\le\frac{n}{2}+N+r}[b,a]_m\cdot a_{r+n-m}\Big)+rk\sum_{a\in B}(b,a)\cdot a_{n+r}\ \ominus$$

But : For any $v\in M$ and $N\gg 0$, the first sum acts trivially on $v$.
Likewise the reordered second sum also acts trivially on $v$
$$\sum_{\frac{n}{2}+N<m\le\frac{n}{2}+N+r} a_{r+n-m}\cdot[b,a]_m$$

$$\ominus\ \frac{1}{2}r\sum_{a\in B}[[b,a],a]_{r+n}+rk\sum_{a\in B}(b,a)\cdot a_{n+r}\ \overset{\text{Corollary 1}}{=}\ \boxed{r\cdot b_{n+r}}\quad ✓$$

as $d([b,a]_m,a_{r+n-m})=0$ due to $([b,a],a)=(b,[a,a])=0$

This completes our proof of $[L_n,b_r]=-r\cdot b_{n+r}$.

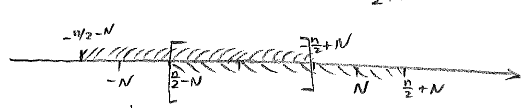
- It remains to prove $\boxed{[L_n,L_m]-(n-m)L_{n+m}=\frac{n^3-n}{12}\delta_{n,-m}\cdot k\sum_{a\in B}(a,a)}$

- First, we note that (1) $\Rightarrow$ $[[L_n,L_m]-(n-m)L_{n+m},b_r]=0$ $\forall b\in\mathfrak{g}, r\in\mathbb{Z}$
- For $n+m\neq 0$: on one hand explicit f-la for $L_0$ together with ↑ implies $[[L_n,L_m]-(n-m)L_{n+m},L_0]=0$.
But on the other hand, $[L_0,b_r]=-r\,b_r \Rightarrow [[L_n,L_m]-(n-m)L_{n+m},L_0]=(n+m)\cdot([L_n,L_m]-(n-m)L_{n+m})$
$\Rightarrow \boxed{[L_n,L_m]-(n-m)L_{n+m}=0 \text{ for } n+m\neq 0.}$
- For $m=-n$:

$$[L_n,L_{-n}]=\Big[\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}a_m a_{n-m},\,L_{-n}\Big]\overset{(1)}{=}\frac{1}{2}\lim_{N\to\infty}\Big(\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}m\cdot a_{m-n}a_{n-m}+\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}(n-m)a_m a_{-m}\Big)$$
$$=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Big(\sum_{|m+\frac{n}{2}|\le N}(m+n)a_m a_{-m}+\sum_{|m-\frac{n}{2}|\le N}(-m+n)a_m a_{-m}\Big)$$

so that

$$[L_n,L_{-n}]-2nL_0=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Big(\sum_{-\frac{n}{2}-N\le m\le-\frac{n}{2}+N}(m+n)a_m a_{-m}+\sum_{\frac{n}{2}-N\le m\le\frac{n}{2}+N}(-m+n)a_m a_{-m}-\sum_{\frac{n}{2}-N\le m\le N-\frac{n}{2}}2n:a_m a_{-m}:\Big)$$




$$=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Big\{\underbrace{\Big(\sum_{-\frac{n}{2}-N\le m<\frac{n}{2}-N}(m+n)a_m a_{-m}\Big)}_{\text{acts by ZERO on any } v\in M \text{ for } N\gg0}+\underbrace{\Big(\sum_{-\frac{n}{2}+N<m\le\frac{n}{2}+N}(n-m)a_{-m}a_m\Big)}_{\text{acts by ZERO on any } v\in M \text{ for } N\gg0}+k(a,a)\cdot\Big(\sum_{1\le m\le-\frac{n}{2}+N}m(m+n)+\sum_{1\le m\le\frac{n}{2}+N}m(n-m)\Big)\Big\}$$

But: It is clear that $\sum_{1\le m\le-\frac{n}{2}+N}m(m+n)+\sum_{1\le m\le\frac{n}{2}+N}m(n-m)$ is independent of $N$, while for $N=\frac{n}{2}$, we get

$$0+\sum_{m=1}^{n}m(n-m)=\frac{n(n+1)}{2}\cdot n-\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)(n-1)}{6}=\frac{n^3-n}{6}$$

Thus : $\boxed{[L_n,L_{-n}]-2nL_0=\frac{n^3-n}{12}\cdot k\sum_{a\in B}(a,a)}$ ✓

• Application 1 : $\mathfrak{g}$-abelian.

In this case $Kil \equiv 0$. Hence, $(\cdot,\cdot)$ must be a nondeg bilinear form and $k \neq 0$.
In that scenario, if $\{a\}_{a\in B}$ - orthonormal basis w.r.t. $k(\cdot,\cdot)$, then $c = k\sum_{a\in B}(a,a) = \dim \mathfrak{g}$.
Thus, we get a Virasoro-action on admissible $\hat{\mathfrak{g}}$-modules of charge $\dim\mathfrak{g}$.
In the simplest case $\dim\mathfrak{g}=1$, we get the construction of Lecture 5 (in the case $M = F_\mu$).

• Application 2 : $\mathfrak{g}$ - simple f.d.

The standard choice of the nondeg. inv. symm. bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ (must be multiple of Kil) is such that the corresponding (induced by $(\cdot,\cdot)$) form on $\mathfrak{g}^*$ satisfies $\boxed{(\alpha,\alpha)=2 \text{ for long roots}}$ or equivalently $(\theta,\theta)=2$ where $\boxed{\theta = \text{maximal root of } \mathfrak{g}}$

Def 3: The dual Coxeter number $h^\vee$ of $\mathfrak{g}$ is defined by $\boxed{h^\vee := 1+(\theta,\rho)}$ $\left(\begin{array}{l}\text{as always}\\ \rho = \frac{1}{2}\sum_{\alpha\in\Delta^+ = \text{positive roots}}\alpha\end{array}\right)$

Table :

| Type | $h^\vee$ |
|---|---|
| $A_n$ | $n+1$ |
| $B_n$ | $2n-1$ |
| $C_n$ | $n+1$ |
| $D_n$ | $2n-2$ |
| $E_6$ | 12 |
| $E_7$ | 18 |
| $E_8$ | 30 |
| $F_4$ | 9 |
| $G_2$ | 4 |

Proposition 1 : $Kil(a,b) = 2h^\vee(a,b)$

▸ Recall the Casimir element $C = \sum_{a\in B'} a^2$, where $B'$ is an orthonormal basis of $\mathfrak{g}$ w.r.t. $(\,,\,)$.
It is central (cf. Lemma 2). Hence it acts by scalar $\gamma_\lambda$ on any irreducible f.d. $\mathfrak{g}$-rep-n $L_\lambda$.
This constant $\gamma_\lambda$ is explicitly given by

$$\boxed{\gamma_\lambda = (\lambda, \lambda+2\rho)}$$

Exercise : Prove this well-known basic fact!

Let us apply this to the adjoint representation, i.e. $\underline{\underline{\lambda=\theta}}$.

Then :

$$\left.\begin{array}{l} Tr_{\mathfrak{g}}(C) = \gamma_\theta\cdot\dim\mathfrak{g} = \gamma_\theta\cdot\sum_{a\in B'}(a,a) \\ \quad\shortparallel \\ Tr_{\mathfrak{g}}\left(\sum_{a\in B'} ad(a)^2\right) = \sum_{a\in B'} Kil(a,a) \\ \text{But : Kil and } (\cdot,\cdot) \text{ are nonzero multiples of each other} \end{array}\right\} \Rightarrow \boxed{Kil(a,b) = \gamma_\theta\cdot(a,b) \;\; \forall a,b\in\mathfrak{g}}$$

It remains to notice

$$\boxed{\gamma_\theta = (\theta,\theta+2\rho) = (\theta,\theta)+(\theta,2\rho) = 2+2(\theta,\rho) = 2h^\vee}$$

which completes the proof ◼

**Corollary 2:** For a simple $\mathfrak{g}$ with the standard $(\cdot,\cdot)$, $k$ is non-critical iff $k \neq -h^\vee$

Note that in this case: $k(\cdot,\cdot) + \frac{1}{2}\mathrm{Kil} = (k+h^\vee)\cdot(\cdot,\cdot)$. Hence, Theorem 1 can be restated:

**Theorem 2:** The Sugawara construction for simple $\mathfrak{g}$ defines a repr-n of Vir on admissible $\hat{\mathfrak{g}}$-modules via

$$\boxed{L_n = \frac{1}{2(k+h^\vee)} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}:}$$

with the central charge

$$\boxed{c = \frac{k}{k+h^\vee} \sum_{a\in B'} (a,a) = \frac{k \dim \mathfrak{g}}{k+h^\vee}}$$

where $\{a\}_{a\in B'}$ – orthonormal basis of $\mathfrak{g}$ w.r.t. $(\cdot,\cdot)$.

**Corollary 3:** Any $\hat{\mathfrak{g}}$-module $M$ realized as a quotient of a Verma $\hat{\mathfrak{g}}$-module $M_\lambda^+$ can be naturally endowed with an internal grading (i.e. made into $\tilde{\mathfrak{g}}$-module) via eigenvalues of $L_0$ (i.e. $d = -L_0$)

$\boxed{Q}$ What happens when $k = -h^\vee$?
critical level

Define

$$\boxed{T_n := \frac{1}{2} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}:}$$

Then our proof of Theorem 1 yields:

**Corollary 4:** $[T_n, a_m] = 0$, $[T_n, T_m] = 0$ $\forall n,m\in\mathbb{Z}$, $a\in\mathfrak{g}$.

In other words, $T_n$ are central elements of a certain completion $U(\hat{\mathfrak{g}})^\wedge$! completion

**Exercise:** Show that Corollary 3 fails at the critical level.

- Recalling $\dagger$: Vir $\circlearrowleft$ which sends $L_k \mapsto L_{-k}$, $C \mapsto C$, we get:

**Proposition 2:** In the setup of Theorem 2, if $M$ is a unitary admissible $\hat{\mathfrak{g}}$-module, then $M$ is a unitary $\mathrm{Vir} \ltimes \hat{\mathfrak{g}}$-module.

▶ Recall $\mathfrak{g} \simeq \mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ with $\mathfrak{g}_\mathbb{R} = \{a\in\mathfrak{g} \mid a^\dagger = -a\}$. Choose a $\mathbb{R}$-basis $\{v_j\}$ of $\mathfrak{g}_\mathbb{R}$ such that $(v_j, v_{j'}) = -\delta_{jj'}$

Set $B' = \{u_j = i\cdot v_j\}$ – a $\mathbb{C}$-basis of $\mathfrak{g}$, such that $u_j^\dagger = u_j$ and $(u_j, u_{j'}) = \delta_{jj'}$ $\Rightarrow$ $B'$ – orthonormal w.r.t. $(,)$.

For $n\neq 0$: $(L_n)^\dagger \overset{1/2(k+h^\vee)}{=} \left(\sum_{m\in\mathbb{Z}} \sum_j (u_j)_m (u_j)_{n-m}\right)^\dagger \overset{\frac{1}{2(k+h^\vee)}}{=} \sum_{m\in\mathbb{Z}} \sum_j (u_j)_{m-n} (u_j)_{-m} = L_{-n}$

For $n=0$: $L_0^\dagger = \frac{1}{2(k+h^\vee)} \sum_j \left((u_j)_0 (u_j)_0 + 2\sum_{m>0} (u_j)_{-m} (u_j)_m\right)^\dagger = L_0$ ■

(Same result also applies when $\mathfrak{g}$ is abelian)

However, for a $\hat{\mathfrak{g}}$-module $M$ ($\mathfrak{g}$-simple) to be unitary, we must have $k \in \mathbb{Z}_{\geq 0}$.
(For $\mathfrak{g} = \mathfrak{sl}_n$, this follows from [Thm 1 + Def 3, Lecture 12], while for general $\mathfrak{g}$ the argument is similar).
Moreover, $k=0$ corresponds to the trivial repr-n. Hence, we may assume $k \in \mathbb{Z}_{\geq 1}$.
But it is easy to see that for such $k$, we have:

$$\boxed{c = \frac{k \dim \mathfrak{g}}{k+h^\vee} \geq 1}$$

But for $c \geq 1$ and $h \geq 0$, we already know unitarity of Virasoro modules $L_{c,h}$.

• $\boxed{Q}$: Can we update the above construction to obtain unitary Virasoro modules with $0<c<1$?

The answer to this question is positive and is provided by the "coset construction".
Suppose $\mathfrak{g} \supset \mathfrak{p}$ are two fin. dim. Lie algebras. Let $(\cdot,\cdot)$ be a $\mathfrak{g}$-inv. form on $\mathfrak{g}$ $\Rightarrow$ its restriction is a $\mathfrak{p}$-inv. form on $\mathfrak{p}$. Then we have $L\mathfrak{g} \supseteq L\mathfrak{p} \rightsquigarrow \hat{\mathfrak{g}} \supseteq \hat{\mathfrak{p}}$. Let $k \in \mathbb{C}$ be non-critical for $\mathfrak{g}$ & $\mathfrak{p}$.
Let $M$ be an admissible $\hat{\mathfrak{g}}$-module at level $k$ $\Rightarrow$ $M$ is also an admissible $\hat{\mathfrak{p}}$-module at level $k$.
$\Rightarrow$ get two Virasoro actions with generators $L_i$ acting via $\{L_i^{\mathfrak{g}}\}$ and $\{L_i^{\mathfrak{p}}\}$ given by Theorem 1, and with central charges $c_{\mathfrak{g}}$ and $c_{\mathfrak{p}}$.

**Theorem 3** (Goddard–Kent–Olive '85) Set $\boxed{L_i := L_i^{\mathfrak{g}} - L_i^{\mathfrak{p}}}$ $\forall i \in \mathbb{Z}$ and $\boxed{c := c_{\mathfrak{g}} - c_{\mathfrak{p}}}$. Then:
(a) $\{L_i\}$ define a Virasoro action on $M$ with central charge $c$
(b) $[L_n, L_m^{\mathfrak{p}}] = 0$ $\forall n,m \in \mathbb{Z}$

• For any $b \in \mathfrak{p}, r \in \mathbb{Z}$: $[L_n^{\mathfrak{g}}, b_r] = -r b_{n+r}$, $[L_n^{\mathfrak{p}}, b_r] = -r b_{n+r}$ (by Theorem 1)

$$\Rightarrow [L_n, b_r] = 0 \quad \forall b \in \mathfrak{p}, r \in \mathbb{Z} \Rightarrow [L_n, L_m^{\mathfrak{p}}] = 0 \Rightarrow \text{part (b)}.$$

• $[L_n, L_m] = [L_n, L_m^{\mathfrak{g}} - L_m^{\mathfrak{p}}] \overset{(b)}{=} [L_n, L_m^{\mathfrak{g}}] = [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{\mathfrak{g}}] \overset{(b)}{=} [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{\mathfrak{p}}]$

$$= (n-m)(L_{n+m}^{\mathfrak{g}} - L_{n+m}^{\mathfrak{p}}) + \frac{n^3-n}{12}\delta_{n,-m}(c_{\mathfrak{g}} - c_{\mathfrak{p}}) = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\cdot c \Rightarrow \text{part (a)}$$

■

• Let us apply this result in the following very particular case:
Let $\mathfrak{a}$ be a simple f.d. Lie alg, $\mathfrak{g} := \mathfrak{a} \oplus \mathfrak{a}$, $\mathfrak{p} = \mathfrak{a}_{\mathrm{diag}} \hookrightarrow \mathfrak{g}$. Let $(\cdot,\cdot)$ be the standard invariant form on $\mathfrak{a}$, which gives rise to the form $(\cdot,\cdot)$ on $\mathfrak{g}$ (as a direct sum).
Let $V', V''$ be admissible $\hat{\mathfrak{a}}$-modules of levels $k', k''$. Assuming $k' \neq -h_{\mathfrak{a}}^\vee \neq k''$, we obtain Virasoro actions $\{L_i'\}, \{L_i''\}$ on $V', V''$ with central charges $c_{\mathfrak{a}}' = \frac{k' \dim \mathfrak{a}}{k'+h^\vee}$, $c_{\mathfrak{a}}'' = \frac{k'' \dim \mathfrak{a}}{k''+h^\vee}$ (for simplicity $h_{\mathfrak{a}}^\vee \rightsquigarrow h^\vee$)
Also consider $V = V' \otimes V''$. It is an admissible $\hat{\mathfrak{g}}$-module at level $k'+k''$ $\Rightarrow$ also admissible $\hat{\mathfrak{p}}$-module of level $k'+k''$.
Then the action $(L_i^{\mathfrak{g}})$ of $\mathrm{Vir} \curvearrowright V$ is given by $\boxed{L_i^{\mathfrak{g}} = L_i' \otimes \mathrm{id} + \mathrm{id} \otimes L_i''}$, while the central charge is $\boxed{c_{\mathfrak{g}} = c_{\mathfrak{a}}' + c_{\mathfrak{a}}'' = \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee}\right)\dim \mathfrak{a}}$. On the other hand, viewing $V$ as $\hat{\mathfrak{p}}$-module we get $\mathrm{Vir} \curvearrowright V$ via $(L_i^{\mathfrak{p}})$ with the central charge $\boxed{c_{\mathfrak{p}} = \frac{k'+k''}{k'+k''+h^\vee}\dim \mathfrak{a}}$

We can summarize the above discussion as follows:

Proposition 3: Applying Theorem 3 in the above context, we obtain $Vir \curvearrowright V' \otimes V''$ with

$$\boxed{\begin{aligned} L_n = & \left(\frac{1}{2(k'+h^\vee)} - \frac{1}{2(k'+k''+h^\vee)}\right) \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}: \otimes 1 \\ & + \left(\frac{1}{2(k''+h^\vee)} - \frac{1}{2(k'+k''+h^\vee)}\right) \sum_{m\in\mathbb{Z}} \sum_{a\in B'} 1 \otimes :a_m a_{n-m}: \\ & - \frac{1}{k'+k''+h^\vee} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} a_m \otimes a_{n-m} \end{aligned}}$$

with the central charge

$$\boxed{c = \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee} - \frac{k'+k''}{k'+k''+h^\vee}\right) \dim \mathfrak{g}}$$

Moreover, the operators $L_n$ commute with $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{diag}$-action.

Example: Let $\mathfrak{g} = \mathfrak{sl}_2$ ($\Rightarrow h^\vee = 2$), $k' = 1$, $k'' = m$. Then $c = 3\left(\frac{1}{3} + \frac{m}{m+2} - \frac{m+1}{m+3}\right)$, i.e

$$\boxed{c = 1 - \frac{6}{(m+2)(m+3)}}$$

these are exactly values of $c$ from "discrete series" of Lecture 11

This allows to construct unitary $Vir$-representations with $c = 1 - \frac{6}{(m+2)(m+3)}$.
But to develop this theory, we need to study further affine Lie alg-s.