Today: Kac-Moody algebras

But we shall start by recalling the basic fact on simple fin. dim. Lie algebras

- Let $\mathfrak{g}$ be a simple fin. dim. Lie algebra over $\mathbb{C}$.

A Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal commutative Lie subalgebra consisting of semisimple elements (it is not unique, but all Cartan subalgebras are conjugate to each other under the action of the corresponding Lie group G s.t. $\mathrm{Lie}(G)=\mathfrak{g}$; hence, we may pick a Cartan subalg. in the beginning)

Then: $\circ$ $\dim \mathfrak{h} =: \mathrm{rk}(\mathfrak{g})$ – rank of $\mathfrak{g}$

$\circ$ $\mathrm{Kil}_{|\mathfrak{h}\times\mathfrak{h}}$ is a nondegenerate symm. bilinear form.

For $\alpha \in \mathfrak{h}^*$, define $\boxed{\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h,x]=\alpha(h)\cdot x \ \forall h \in \mathfrak{h}\} \subseteq \mathfrak{g}}$

Then: $\circ$ $\mathfrak{g}_0 = \mathfrak{h}$

$\circ$ $\Delta := \{\alpha \overset{\neq 0}{\in} \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$ is a finite set, called the root system of $\mathfrak{g}$

$\circ$ $\boxed{\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha}$

$\circ$ $\forall \alpha \in \Delta$: $\mathfrak{g}_\alpha$ is 1-dim, hence, $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$ for some choice of a basis vector e_α

Fix $\bar{h} \in \mathfrak{h}$ s.t. $\alpha(\bar{h}) \in \mathbb{R}\setminus\{0\}$ $\forall \alpha$ and set $\boxed{\Delta_\pm := \{\alpha \in \Delta \mid \pm\alpha(\bar{h}) > 0\}}$ so that $\Delta = \Delta_+ \amalg \Delta_-$, $\Delta_- = -\Delta_+$

(Exercise: Show that such $\bar{h}$ exists)

Δ_+ = positive roots of $\mathfrak{g}$, Δ_- = negative roots of $\mathfrak{g}$

Then, we get the triangular decomposition

$$\boxed{\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \mathfrak{n}_\pm := \bigoplus_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha \text{ – Lie subalg-s}}$$

It is known that $\mathfrak{h}^*$ has a basis of simple roots $\{\alpha_1, \ldots, \alpha_{n=\mathrm{rk}(\mathfrak{g})}\} \subset \Delta_+$ (cannot be written as sums of more than 1 el-s of Δ_+) such that any $\alpha \in \Delta_+$ may be written as $\alpha = \sum_{i=1}^n k_i \alpha_i$ in a unique way with $k_i \in \mathbb{Z}_{\geq 0}$.

Then: $\circ$ If $\alpha+\beta \notin \Delta \cup \{0\} \Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

$\circ$ If $\alpha+\beta \in \Delta \Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$

$\circ$ If $\alpha+\beta = 0 \Rightarrow \underset{\neq 0}{[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]} \subseteq \mathfrak{h}$

$\circ$ For every $\alpha \in \Delta_+$, pick $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, $f_\alpha \in \mathfrak{g}_{-\alpha}\setminus\{0\}$ (unique up scalars of $\mathbb{C}^\times$), in particular, $e_i := e_{\alpha_i}$, $f_i := f_{\alpha_i}$

Set $h_i := [e_i, f_i] \in \mathfrak{h}$; more generally h_α ($\alpha \in \Delta_+$) denotes a nonzero basis vector of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

$\circ$ It is possible to normalize e_i, f_i, so that $\{e_i, h_i, f_i\}$ form an $\mathfrak{sl}_2$-triple:

$$\boxed{[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad [e_i, f_i] = h_i}$$

Prop 1: (1) $\{h_1, \ldots, h_n\}$ form a basis of $\mathfrak{h}$

(2) $\mathfrak{g}$ is generated (as a Lie algebra) by $\{e_i, h_i, f_i\}_{i=1}^n$

(3) The following relations hold (BUT these are not all the defining rel-s!):

$\boxed{[h_i, h_j] = 0, \ [h_i, e_j] = \alpha_j(h_i)\, e_j, \ [h_i, f_j] = -\alpha_j(h_i)\cdot f_j, \ [e_i, f_j] = \delta_{ij} h_i}$ ← these rel-s are obvious!

But if we consider a Lie algebra $\tilde{\mathfrak{g}}$ gen-d by $\{e_i, h_i, f_i\}_{i=1}^n$ with the defining rel-s as in Prop 1(3), then $\dim(\tilde{\mathfrak{g}}) \underset{\text{unless } \mathrm{rk}(\mathfrak{g})=1}{=} \infty$ and $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ with a nontrivial kernel (generated by the Serre relations)

Let $(\cdot,\cdot)$ be the standard invariant bilinear form on $\mathfrak{g}$ as in Lecture 13, so that $(\,,\,) = \frac{Kil}{2h^\vee}$.
It gives a nondeg. pairing on $\mathfrak{h}$ $\Rightarrow$ identification $\mathfrak{h} \simeq \mathfrak{h}^*$ $\Rightarrow$ nondeg. pairing on $\mathfrak{h}^*$.
Identifying $\mathfrak{h} \simeq \mathfrak{h}^*$ as above, $\boxed{h_i \mapsto \alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}}$ Set $\boxed{a_{ij} := \alpha_j(h_i) = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}}$ ← Here it is clearly irrelevant which invariant form $(\,,\,)$ is used!

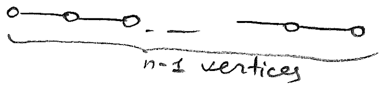
Thus: $\boxed{[h_i, e_j] = a_{ij}\cdot e_j, \quad [h_i, f_j] = -a_{ij} f_j}$

Prop 2: The following properties of the Cartan matrix $A = (a_{ij})_{i,j=1}^n$ hold:
1) $a_{ii} = 2$
2) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ and $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.
3) A is indecomposable, i.e. it cannot be conjugated by a permutation to $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$.
4) A is positive, i.e. $\exists$ diagonal matrix D with positive diagonal entries, s.t. DA – symm. & positive definite

The following result is proved in the fin. dim. Lie alg. course:

Thm 1: (1) Matrix A satisfies properties (1–4) iff A is a Cartan matrix of a simple f.d. Lie algebra.
(2) A complete classification of such matrices is via their Dynkin diagrams
(defined as graphs with vertex set $\{1, 2, \ldots, n\}$ and the following rules for edges:
- if $a_{ij} = 0 \Rightarrow$ no edges b/w i & j
- if $a_{ij} = -1 = a_{ji} \Rightarrow$ i & j are connected by exactly 1 unoriented edge i — j
- if $a_{ij} = -2$, $a_{ji} = -1 \Rightarrow$ we draw $i \Leftarrow j$ (2 edges b/w i & j directed towards i)
- if $a_{ij} = -3$, $a_{ji} = -1 \Rightarrow$ we draw $i \Lleftarrow j$ (3 edges b/w i & j directed towards i)

These are all possibilities due to positivity of $\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$)



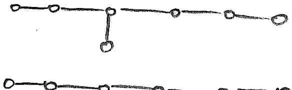
$A_{n-1} = \mathfrak{sl}_n$, $n \geq 2$

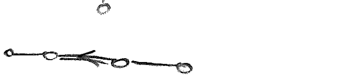

$B_n = \mathfrak{so}_{2n+1}$, $n \geq 2$


$C_n = \mathfrak{sp}_{2n}$, $n \geq 3$


$D_n = \mathfrak{so}_{2n}$, $n \geq 4$


E_6


E_7


E_8

Rmk: From the f-la for a_{ij} it is clear that oriented edges are oriented towards shorter roots!


F_4

G_2

Thm 2: (1) Let $i \neq j$. Then, the following Serre relations hold in $\mathfrak{g}$:

$$\boxed{(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0, \quad (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0}$$

(2) The kernel of $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ is generated by the Serre relations

Proof of Thm 2(1)

In the adjoint representation $\mathfrak{g} \overset{\mathrm{ad}}{\curvearrowright} \mathfrak{g}$, elt $f_j \in \mathfrak{g}$ is a highest weight vector w.r.t. $\mathfrak{sl}_2$-triple: (e_i, h_i, f_i)

- $[e_i, f_j] = 0$ as $i \neq j$
- $[h_i, f_j] = -a_{ij} \cdot f_j$, where $-a_{ij} \in \mathbb{Z}_{\geq 0}$

But the $\mathfrak{sl}_2$-theory then implies that $(\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0$ (it has weight $-a_{ij} - 2(1-a_{ij}) = a_{ij} - 2$). ∎

Now that we have recalled the classical result on simple f.d. Lie algebras, let us go to the more general setting of contragredient Lie alg-s.

- **Contragredient Lie algebras**

Let A be an $n \times n$ matrix of complex numbers. Let Q be a free abelian gp of rank n, basis: $\{\alpha_i\}_{i=1}^n$
root lattice $\quad Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \ldots \oplus \mathbb{Z}\alpha_n$

Def 1: A contragredient Lie algebra corresponding to A is a (Q-graded) $\mathbb{C}$-Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ generated (as a Lie algebra) by $\{e_i, h_i, f_i\}_{i=1}^n$ satisfying 3 conditions:

(1) these el-s satisfy relations of Prop 1(3):
$[h_i, h_j] = 0$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$, $[e_i, f_j] = \delta_{ij} \cdot h_i$

(2) $\mathfrak{g}_0$ has a basis $\{h_i\}_{i=1}^n$, $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$.

(3) Any nonzero Q-graded ideal has a nonzero intersection with $\mathfrak{g}_0 =: \mathfrak{h}$.

Rmk 1: If $\mathfrak{g}$ is simple, condition (3) is immediate.

Cor 1: Simple f.d. Lie algebras are contragredient

Thm 3: If A is a complex $n \times n$ matrix, then there exists a unique contragredient Lie alg. $\mathfrak{g}$ corresponding to A.

As before, consider the Lie algebra $\tilde{\mathfrak{g}}(A)$ generated by $\{e_i, h_i, f_i\}_{i=1}^n$ subject to the defining relations of Def 1(1).

Claim: $\boxed{\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+}$, where $\mathfrak{h}$ has a basis $\{h_i\}_{i=1}^n$
$\tilde{\mathfrak{n}}_+$ is the free Lie algebra generated by $\{e_i\}_{i=1}^n$
$\tilde{\mathfrak{n}}_-$ is the free Lie algebra generated by $\{f_i\}_{i=1}^n$.

Let us first explain how this claim allows us to prove Thm 3.

The Lie algebra $\tilde{\mathfrak{g}}(A)$ obviously satisfies conditions (1,2) of Def 1.
Let I be the sum of all Q-graded ideals in $\tilde{\mathfrak{g}}(A)$ that have zero intersection with $\mathfrak{h}$.
Clearly $I = I_+ \oplus I_-$ with $I_\pm := I \cap \tilde{\mathfrak{n}}_\pm$.

Set $\boxed{\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/I}$

By the very construction, $\mathfrak{g}(A)$ satisfies (3). It clearly satisfies (1). As for (2):
the fact that $\mathfrak{h}$ has a basis $\{h_i\}_{i=1}^n$ is obvious, while the images of e_i, f_i in $\mathfrak{g}(A)$ are nonzero (as if $e_i \in I \Rightarrow e_i \in I'$ - Q-graded ideal non-intersecting $\mathfrak{h}$ $\Rightarrow h_i = [e_i, f_i] \in I' \cap \mathfrak{h} \Rightarrow$ ↯)

$\underline{\underline{\text{So}}}$: $\mathfrak{g}(A)$ satisfies all 3 conditions of Def 1.

If $\mathfrak{g}'$ is another Lie alg. satisfying same conditions $\Rightarrow$ $\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}'$ factors through $\mathfrak{g}(A) \twoheadrightarrow \mathfrak{g}'$. But Kernel is a graded ideal that has zero intersection with $\mathfrak{h}$

$\underline{\underline{\text{So}}}$: $\mathfrak{g}(A) \xrightarrow{\sim} \mathfrak{g}'$. $\Rightarrow$ contragredient alg. associated with A is unique up to isom.

We shall prove Claim next time!