— LECTURE 15 —

• Last time: Introduced the notion of contragredient Lie algebras associated to any $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

Thm1: For any A, there is a unique (up to an isom) contragredient Lie algebra $\mathfrak{g}(A)$.

Let $\tilde{\mathfrak{g}}(A)$ be a "larger" Lie algebra generated by $\{e_i, h_i, f_i\}_{i=1}^n$ with the defining relations:

$$\boxed{[h_i, h_j]=0, \quad [h_i, e_j]=a_{ij}\cdot e_j, \quad [h_i, f_j]=-a_{ij}\cdot f_j, \quad [e_i, f_j]=\delta_{ij}\cdot h_i}$$

Last time, we proved the above theorem based on the following result:

Thm2: Let $\tilde{\mathfrak{h}}, \tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_- \subseteq \tilde{\mathfrak{g}}(A)$ be the Lie subalgebras generated by $\{h_i\}_{i=1}^n$, $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$, resp. Then:

a) $\tilde{\mathfrak{g}}(A) \simeq \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$

b) $\tilde{\mathfrak{h}}$ has the basis $\{h_i\}_{i=1}^n$

$\tilde{\mathfrak{n}}_+ \simeq \mathrm{FreeLie}(\{e_i\}_{i=1}^n)$ — free Lie alg. on $\{e_i\}$

$\tilde{\mathfrak{n}}_- \simeq \mathrm{FreeLie}(\{f_i\}_{i=1}^n)$ — free Lie alg. on $\{f_i\}$

Remark1: (a) Given a set X, the free Lie alg. on X is a unique (up to isom.) Lie alg. $\mathrm{FreeLie}(X)$ together with a set map $\imath: X \to \mathrm{FreeLie}(X)$, s.t. for any Lie alg. L and a set map $f: X \to L$ $\exists!$ Lie alg. homom. $\mathrm{FreeLie}(X) \xrightarrow{F} L$ satisfying $F\circ\imath = f$.

(b) Given a vector space V, the free Lie alg. on V is a unique (up to isom.) Lie alg. $\mathrm{FreeLie}(V)$ together with a linear map $\imath: V \to \mathrm{FreeLie}(V)$, s.t. for any Lie alg. L and a linear map $f: V \to L$ $\exists!$ Lie alg. homom. $\mathrm{FreeLie}(V) \xrightarrow{F} L$ satisfying $F\circ\imath = f$

(c) If V is a v. space with basis X, then $\mathrm{FreeLie}(V) \simeq \mathrm{FreeLie}(X)$.

Exercise1: (a) Show that maps $\imath$ are injective in both cases

(b) Prove $U(\mathrm{FreeLie}(X)) \simeq k\langle X\rangle$ — free algebra on the set X (k - ground field)

(c) Prove $U(\mathrm{FreeLie}(V)) \simeq T(V)$ — tensor algebra on the vector space V.

Proof of Thm2

• First, we note that the sum $\tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ is indeed direct! One way to see this is to consider $\mathbb{Z}$-grading with $\deg(e_i)=1$, $\deg(h_i)=0$, $\deg(f_i)=-1$ (such a grading exists since all defining rel-s are homogenious), but then $\deg(\tilde{\mathfrak{n}}_-)<0$, $\deg(\tilde{\mathfrak{h}})=0$, $\deg(\tilde{\mathfrak{n}}_+)>0$.

• We actually have the equality $\tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+ = \tilde{\mathfrak{g}}(A)$. As $\tilde{\mathfrak{g}}(A)$ is generated by $\{e_i, h_i, f_i\}$, to prove the latter, it suffices to verify that $\tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ is stable under $\mathrm{ad}(h_i)$, $\mathrm{ad}(e_i)$, $\mathrm{ad}(f_i)$ which easily follows from the defining relations.

• It remains to prove part (b). For the latter, we use the standard argument:

Exercise2: (a) Construct an action of $\tilde{\mathfrak{g}}$ on $U(\mathfrak{h} \ltimes \mathrm{FreeLie}(\{e_i\}_{i=1}^n))$, $\mathfrak{h}$ - v.space w/ basis $\{h_i\}_{i=1}^n$.

Hint: You may think of it as a universal Verma module over $\tilde{\mathfrak{g}}(A)$.

(b) Deduce that $\tilde{\mathfrak{n}}_+ \simeq \mathrm{FreeLie}(\{e_i\})$, $\tilde{\mathfrak{h}} \simeq \mathfrak{h}$.

(c) Use the similar argument or automorphism $e_i \mapsto f_i$, $f_i \mapsto e_i$, $h_i \mapsto -h_i$ to prove $\tilde{\mathfrak{n}}_- \simeq \mathrm{FreeLie}(\{f_i\})$ ■

So: $A \in Mat_{n\times n}(\mathbb{C}) \rightsquigarrow$ uniquely defined (up to isom.) contragredient $\mathfrak{g}(A)$.

Lemma 1: (a) If $A' = \sigma A \sigma^{-1}$ with σ - permutation matrix, then $\mathfrak{g}(A') \simeq \mathfrak{g}(A)$

(b) If $A' = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$, then $\mathfrak{g}(A') \simeq \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$.

Exercise 3: Prove this simple Lemma

For general A, not much theory is known for $\mathfrak{g}(A)$, but we will primarily focus on the cases when the theory is well-established.

Def 1: $A \in Mat_{n\times n}(\mathbb{C})$ is called a generalized Cartan matrix if:

1) $a_{ii} = 2 \quad \forall\, 1 \le i \le n$.

2) For $i \ne j$: $a_{ij} \in \mathbb{Z}_{\le 0}$ and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$

3) A is symmetrizable, i.e. $\exists$ diagonal matrix D with entries in $\mathbb{R}_{>0}$ s.t. $(DA)^T = DA$.

Rmk 2: A - Cartan matrix iff A - generalized Cartan & DA - positive ($DA > 0$).

Example 1: For $m \in \mathbb{Z}_{>0}$, $A = \begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix}$ is a generalized Cartan matrix (take $D = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$)

For $m = 1$: $\mathfrak{g}(A) \simeq \mathfrak{sl}_3$

$m = 2$: $\mathfrak{g}(A) \simeq \mathfrak{sp}_4 \simeq \mathfrak{so}_5$

$m = 3$: $\mathfrak{g}(A) \simeq \mathfrak{g}_2$ - of type G_2.

$m = 4$: twisted version of $\widehat{\mathfrak{sl}}_2$ - type $A_2^{(2)}$.

$m \ge 5$: BIG (has an exponential growth)

Def 2: A (symmetrizable) Kac-Moody algebra is a Lie alg. of the form $\mathfrak{g}(A)$ for a generalized Cartan A.

Thm 3 (Gabber-Kac): For a Kac-Moody algebra $\mathfrak{g}(A)$, the ideal $I = \mathrm{Ker}(\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A))$ is generated by the Serre rel-s $(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0 = (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j \quad \forall i \ne j$.

Partial Proof of Thm 3

Let us prove that Serre rel-s hold in $\mathfrak{g}(A)$. Fix $i \ne j$ and consider an element $(\mathrm{ad}\, f_i)^{1-a_{ij}} f_j \in \tilde{\mathfrak{n}}_- \subseteq \tilde{\mathfrak{g}}(A)$. It suffices to show that it commutes with all $\{e_k\}_{k=1}^n$, as then the ideal generated by this element belongs to $\tilde{\mathfrak{n}}_-$ $\Rightarrow$ has zero intersection with $\tilde{\mathfrak{h}}$ $\Rightarrow$ annihilated under the projection $\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A)$.

Case 1: $k \ne i, j$.

As $[e_k, f_i] = 0 = [e_k, f_j] \Rightarrow [e_k, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j] = 0$. ✓

Case 2: $k = j$

As $[e_j, f_i] = 0$, $[e_j, f_j] = h_j \Rightarrow [e_j, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j] = (\mathrm{ad}\, f_i)^{1-a_{ij}} h_j$.

If $a_{ij} < 0 \Rightarrow [f_i, [f_i, h_j]] = [f_i, a_{ji} f_i] = 0 \Rightarrow (\mathrm{ad}\, f_i)^{1-a_{ij}} h_j = 0$ ✓

If $a_{ij} = 0 \Rightarrow [f_i, h_j] = 0$

Case 3: $k = i$

Consider $\mathfrak{sl}_2^{(i)} := \langle e_i, h_i, f_i \rangle$. As $[e_i, f_j] = 0$, $[h_i, f_j] = -a_{ij} f_j \xRightarrow{\mathfrak{sl}_2\text{-theory}} (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j$ - singular vector for $\mathfrak{sl}_2^{(i)}$

$\Rightarrow [e_i, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j] = 0$

Def 3: A generalized Cartan matrix A is called *affine* iff $DA \geq 0$, but $DA \not> 0$

Rmk 3: In particular, $\det(A)=0$.

Def 4: If A is an affine generalized Cartan matrix, then $\mathfrak{g}(A)$ is called *affine Kac-Moody algebra*

Let us now explain how this is consistent with our previous definition of affine $\hat{\mathfrak{g}}$! (so-called *untwisted* affine KM alg-s)

- Let $\mathfrak{g}$ be a f.d. simple $\mathbb{C}$-Lie alg., in particular, $\mathfrak{g} \simeq \mathfrak{g}(A)$ with A - Cartan.
  Recall the loop algebra $L\mathfrak{g}$ and its 1-dim central extension $\hat{\mathfrak{g}}$.

Thm 4: $\hat{\mathfrak{g}}$ is an affine Kac-Moody algebra with an affine Cartan matrix $\hat{A}$ of the form
$$\hat{A} = \left(\begin{array}{c|ccc} 2 & * & \cdots & * \\ \hline * & & & \\ \vdots & & A & \\ * & & & \end{array}\right) \text{ - indecomposable}$$

Pick a Cartan subalg. $\mathfrak{h} \subseteq \mathfrak{g}$, let $r := \mathrm{rk}(\mathfrak{g}) = \dim \mathfrak{h}$. Let $\{e_i, h_i, f_i\}_{i=1}^{r}$ be the Chevalley generators of $\mathfrak{g}$.
Recall that θ = maximal root. We define e_0, f_0, h_0 via:
$$\boxed{e_0 = f_\theta \cdot t,\ f_0 = e_\theta \cdot t^{-1},\ h_0 = K - h_\theta}$$
where $\langle e_\theta, h_\theta, f_\theta \rangle$ form an $\mathfrak{sl}_2$-triple corresponding to θ.

Thus, we obtain elements $\{e_i, h_i, f_i\}_{i=0}^{r}$.

* First, we claim that these elements generate $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}\cdot K$.
They clearly generate $\mathfrak{g} \oplus \mathbb{C}\cdot K$. Also note that $\mathfrak{g}\cdot t\mathbb{C}[t]$ (resp. $\mathfrak{g}\cdot t^{-1}\mathbb{C}[t^{-1}]$) is generated by $\mathfrak{g}t$ (resp. $\mathfrak{g}t^{-1}$) as $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. But as $\mathfrak{g}$-simple, $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}t$ - irreducible with the lowest wt vector $= f_\theta \cdot t \Rightarrow \mathfrak{g}t$ is generated by these el-s (similarly also $\mathfrak{g}t^{-1}$ is generated).

* Let's verify relations now.

- The relation $[h_i, h_j] = 0$ $\forall i,j \in \{0,1,\dots,r\}$ is clear.
- For $i \neq 0$: $[h_0, e_i] = [K - h_\theta, e_i] = \underbrace{-\alpha_i(h_\theta)}_{a_{0i} \in \mathbb{Z}_{\leq 0}} \cdot e_i$, where $a_{0i} := -\alpha_i(h_\theta) = -(\alpha_i, \theta^\vee = \frac{2\theta}{(\theta,\theta)} \overset{\downarrow}{=} \theta) = -(\alpha_i, \theta)$ - integer and it is ≤ 0 as θ-maximal root ($s_{\alpha_i}\theta \leq \theta$, s_{α_i}-reflection)
  (due to the normalization $(\theta,\theta)=2$)
  For $i=0$: $[h_0, e_0] = [K - h_\theta, f_\theta t] = \theta(h_\theta)\cdot f_\theta t = (\theta, \theta^\vee)\cdot f_\theta t = 2\cdot f_\theta t = 2e_0$.
  For $i \neq 0$: $[h_i, e_0] = [h_i, f_\theta t] = -\theta(h_i)\cdot f_\theta t = \underbrace{-(\alpha_i^\vee, \theta)}_{a_{i0} \in \mathbb{Z}_{\leq 0}} \cdot f_\theta t$, where $a_{i0} := -(\frac{2\alpha_i}{(\alpha_i,\alpha_i)}, \theta) = -(\alpha, \theta)\cdot \underbrace{\frac{(\theta,\theta)}{(\alpha_i,\alpha_i)}}_{\in\{1,2,3\}} \in \mathbb{Z}_{\leq 0}$
- The rel-ns $[h_i, f_j] = -a_{ij} \cdot f_j$ are verified same way
- $[e_0, f_0] = [f_\theta t, e_\theta t^{-1}] = -h_\theta + K\cdot(f_\theta, e_\theta) = -h_\theta + K$, as $2\cdot(f_\theta, e_\theta) \overset{\shortparallel}{=} ([f_\theta, h_\theta], e_\theta) = -(h_\theta, [f_\theta, e_\theta]) = (h_\theta, h_\theta) = (\theta^\vee, \theta^\vee) = (\theta,\theta) = 2$
  $[e_0, f_i] = [f_\theta t, f_i] = 0$ for $i \neq 0$ as θ = max root
  $[e_i, f_0] = [e_i, e_\theta t^{-1}] = 0$ for $i \neq 0$ — // —

Thus, all the required relations do hold!

Note: To make sense of this, we should actually work with $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$ whereas $\delta(d)=1$!

*Set $\boxed{\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}\cdot K}$. Let $\alpha_0 := \delta - \theta \in \hat{\mathfrak{h}}^*$, where $\delta|_{\hat{\mathfrak{h}}} = 0$ and $\alpha(K) = 0$ $\forall \alpha \in \mathfrak{h}^*$.

Then: $[h, e_0] = \alpha_0(h) e_0$ and $[h, f_0] = -\alpha_0(h)\cdot f_0$ $\forall h \in \hat{\mathfrak{h}}$
(indeed $a_{i0} = -\theta(h_i) = \alpha_0(h_i)$ for $1 \leq i \leq r$, while $\alpha_0(h_0) = (\delta - \theta)(h_0) = -\theta(h_0) = -\theta(K - h_\theta) = (\theta, h_\theta) = 2$)

Set $\boxed{\hat{Q} := \mathbb{Z}\alpha_0 \oplus \underbrace{\mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_r}_{Q} = \mathbb{Z}\alpha_0 \oplus Q}$. Then $\hat{\mathfrak{g}}$ is $\hat{Q}$-graded with $\deg(e_i) = \alpha_i$, $\deg(h_i) = 0$, $\deg(f_i) = -\alpha_i$ for $i \in \{0,1,\dots,r\}$.

This $\hat{Q}$-grading is explicitly defined by setting $\deg(K) = 0$, $\deg(x t^n) = \overbrace{\deg_Q(x)}^{\in Q} + n\delta$ $\forall x \in \mathfrak{g}, n \in \mathbb{Z}$. This explains its existence! Note that $\hat{\mathfrak{g}}_0 = \hat{\mathfrak{h}}$, $\hat{\mathfrak{g}}_{\alpha_i} = \mathbb{C}e_i$, $\hat{\mathfrak{g}}_{-\alpha_i} = \mathbb{C}f_i$ for $i \in \{0,1,\dots,r\}$.

(Continuation of proof of Theorem 4)

It remains to show that if $I \subseteq \hat{\mathfrak{g}}$ is a $\hat{Q}$-graded ideal s.t. $I \cap \hat{\mathfrak{h}} = 0$, then $I = 0$.
Assume not. Let $\bar{I} \subseteq L\mathfrak{g} = \overset{\circ}{\hat{\mathfrak{g}}}/\mathbb{C}K$ be the image of I under the natural projection.
As $I \neq 0$, $K \notin I \Rightarrow \bar{I} \neq 0$. Also note that $\bar{I}$ is $\hat{Q}$-graded. But $\hat{Q}$degrees appearing in $\mathfrak{g}t^n, \mathfrak{g}t^m \subseteq L\mathfrak{g}$ with $n \neq m$ have no common values. So: $\exists\, a \in \overset{\circ}{\mathfrak{g}}$, $n \in \mathbb{Z}$ such that $at^n \in \bar{I}$.
But $\mathfrak{g}$-simple $\Rightarrow$ in the adjoint repr-n $a \neq 0$ generates the entire $\mathfrak{g}$ $\Rightarrow$ at^n generates $L\mathfrak{g} \Rightarrow \bar{I} = L\mathfrak{g}$ $\Rightarrow \mathfrak{h} \in \bar{I} \Rightarrow$ Contradiction!

Rmk 4: a) If $\theta = \sum_{i=1}^{r} a_i \cdot \alpha_i$, $a_i \in \mathbb{Z}_{\geq 0}$ (actually $a_i \in \mathbb{Z}_{>0}$!), then $\delta = \alpha_0 + \sum_{i=1}^{r} a_i \alpha_i$ and $(\delta, \alpha_i) \overset{\forall i}{=} 0$ implies that the linear combination of columns of $\hat{D}\hat{A}$ with coeff-s $\{1, a_1, \ldots, a_r\}$ is zero $\Rightarrow$ $\hat{A}$ - degenerate.
BUT its kernel is exactly 1-dim!

b) The fact that $\hat{A}$ is indecomposable may be verified case-by-case (see discussion below).

c) Note that $\hat{A}$ is symmetrizable!
Indeed $A = (a_{ij})$ is symmetrizable with $D = \mathrm{diag}\left(\frac{(\alpha_i,\alpha_i)}{2}\right)_{i=1}^{r}$ as $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$
Set $\hat{D} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$. As $a_{i0} = a_{0i} \cdot \frac{(\theta,\theta)}{(\alpha_i,\alpha_i)} = a_{0i} \cdot \frac{2}{(\alpha_i,\alpha_i)} \Rightarrow \hat{D}\hat{A}$ - symmetric.

d) $\hat{D}\hat{A} \geq 0$ as its entries equal $\{(\alpha_i,\alpha_j)\}_{i,j=0}^{r}$ and the pairing $(\ ,\ )$ is positive on $\mathfrak{h}^*$, while $\hat{\mathfrak{h}}^* \longrightarrow \mathfrak{h}^*$ $(\alpha_0 \mapsto -\sum_{i=1}^{r} a_i \alpha_i,\ \alpha_i \mapsto \alpha_i)$ has 1-dim kernel. (We use the fact that $(\alpha_0,\alpha_j) = -(\theta,\alpha_j)$ $\Rightarrow b_0\alpha_0 + b_1\alpha_1 + \ldots + b_r\alpha_r$ - in kernel iff it is a multiple of $\delta = \alpha_0 + \sum a_i \alpha_i$)

! The theory of affine Kac-Moody algebras $\hat{\mathfrak{g}}$ (associated to $\mathfrak{g}$) a.k.a. "affinizations of $\mathfrak{g}$" is the most rich exactly due to the fact that they admit two realizations:
- Kac-Moody realization
- loop realization

Let us now draw the corresponding Dynkin diagrams:

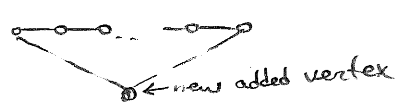
1) $\mathfrak{g} = \mathfrak{g}(A_{n-1}) = \mathfrak{sl}_n$, $n \geq 2$ $\quad A_{n-1}: \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$

$n-1$ vertices

$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{A}_{n-1} = A_{n-1}^{(1)}) = \widehat{\mathfrak{sl}}_n$

$n=2 \rightsquigarrow A_{n-1}^{(1)} = A_1^{(1)}: \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

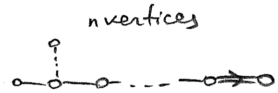
$n>2 \rightsquigarrow A_{n-1}^{(1)}: \begin{pmatrix} 2 & -1 \; 0 \ldots 0 \; -1 \\ \begin{matrix} -1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{matrix} & A_{n-1} \end{pmatrix}$



2) $\mathfrak{g} = \mathfrak{g}(B_n) = \mathfrak{so}_{2n+1}$, $n \geq 3$ $\quad B_n: \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ 0 & & & -2 & 2 \end{pmatrix}$

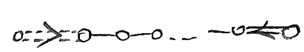
n vertices

$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{B}_n = B_n^{(1)}) = \widehat{\mathfrak{so}}_{2n+1}$ $\quad B_n^{(1)}: \begin{pmatrix} 2 & 0 \; -1 \; 0 \\ \begin{matrix} 0 \\ -1 \\ 0 \end{matrix} & B_n \end{pmatrix}$

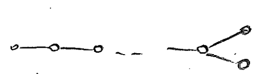


3) $\mathfrak{g} = \mathfrak{g}(C_n) = \mathfrak{sp}_{2n}$, $n \geq 2$ $\quad C_n: \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -2 \\ 0 & & & -1 & 2 \end{pmatrix}$

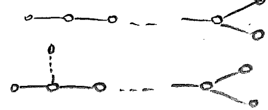
$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{C}_n = C_n^{(1)}) = \widehat{\mathfrak{sp}}_{2n}$ $\quad C_n^{(1)}: \begin{pmatrix} 2 & -1 \; 0 \\ \begin{matrix} -2 \\ 0 \end{matrix} & C_n \end{pmatrix}$



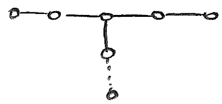
4) $\mathfrak{g} = \mathfrak{g}(D_n) = \mathfrak{so}_{2n}$, $n \geq 4$ $\quad D_n: \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & 2 & -1 & -1 \\ & & -1 & 2 & 0 \\ 0 & & -1 & 0 & 2 \end{pmatrix}$

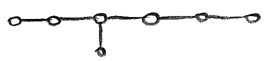


$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{D}_n = D_n^{(1)}) = \widehat{\mathfrak{so}}_{2n}$ $\quad D_n^{(1)}: \begin{pmatrix} 2 & 0 \; -1 \; 0 \\ \begin{matrix} 0 \\ -1 \\ 0 \end{matrix} & D_n \end{pmatrix}$

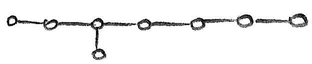


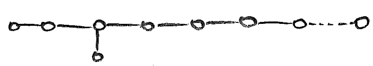
5) $\mathfrak{g} = \mathfrak{g}(E_6)$ 

$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{E}_6 = E_6^{(1)})$ 

6) $\mathfrak{g} = \mathfrak{g}(E_7)$ 

$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{E}_7 = E_7^{(1)})$ 

7) $\mathfrak{g} = \mathfrak{g}(E_8)$ 

$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{E}_8 = E_8^{(1)})$ 

8) $\mathfrak{g} = \mathfrak{g}(F_4)$ 

$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{F}_4 = F_4^{(1)})$ 

9) $\mathfrak{g} = \mathfrak{g}(G_2)$ 

$\hat{\mathfrak{g}} = \mathfrak{g}(\hat{G}_2 = G_2^{(1)})$ 

Exercise 4: (a) Prove above formulas 1)–4) for affinizations of classical Lie alg-s

(b)* Prove above formulas 5)–9) for affinizations of exceptional simple Lie alg-s.

Rmk 5: Besides the above untwisted affine Kac-Moody algebras, there also exist the twisted ones. However, we shall skip their theory right now.