— LECTURE 16 —

*Last time: • $A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \rightsquigarrow$ unique (up to isom.) contragredient Lie algebra $\mathfrak{g}(A)$.

• A-generalized Cartan ($a_{ii}=2$, $a_{ij}\in\mathbb{Z}_{\le 0}$ ($i\ne j$), $a_{ij}=0 \Leftrightarrow a_{ji}=0$, A-symmetrizable) $\Rightarrow \mathfrak{g}(A)$ - Kac-Moody.

• If $\mathfrak{g}(A)$ - Kac-Moody, then $\{\mathrm{ad}(e_i)^{1-a_{ij}}e_j,\ \mathrm{ad}(f_i)^{1-a_{ij}}f_j \mid i\ne j\} \subseteq \mathrm{Ker}(\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A))$

Thm (Gabber – Kac '81): The above el-s actually generate the kernel.

• A - affine iff its symmetrization $DA \geq 0$ (but $DA \not> 0$) $\Rightarrow \mathfrak{g}(A)$ - affine Kac-Moody algebra

• $\mathfrak{g}$-simple f.dim. $\Rightarrow \hat{\mathfrak{g}}$ - affine Kac-Moody with $e_0 = f_\theta t$, $f_0 = e_\theta t^{-1}$, $h_0 = K - h_\theta$.

Def 1: The roots of $\mathfrak{g}(A)$ are el-s of the set $\Delta := \{\alpha \in Q\setminus\{0\} \mid \mathfrak{g}_\alpha \neq 0\}$.

Rmk 1: (a) We have $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm$ is gen-d by $\{e_i\}$, $\{f_i\}$

(b) The existence of automorphism $\tilde{\mathfrak{g}}(A) \circlearrowleft$ sending $e_i \mapsto f_i$, $f_i \mapsto e_i$, $h_i \mapsto -h_i$ gives rise to analogous automorphism of $\mathfrak{g}(A)$.

(c) Using autom. of (b), we see $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$

(d) For positive $\alpha = \sum_{i=1}^n k_i\alpha_i$ ($k_i \in \mathbb{Z}_{\ge 0}$), the subspace $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ is spanned by $[e_{i_1},[e_{i_2},\ldots,[e_{i_{r-1}},e_{i_r}]\ldots]]$ where each e_i ($1\le i\le n$) occurs k_i times

(e) Due to (d), $\dim(\mathfrak{g}_\alpha) < \infty$ $\forall \alpha\in\Delta$

Rmk 2: (a) For the case of $\hat{\mathfrak{g}}$ ($\mathfrak{g}$-simple f.dim) which is $\hat{Q} = \bigoplus_{i=1}^{\tilde{n}} \mathbb{Z}\alpha_i$ -graded via $\deg(e_i) = \alpha_i = -\deg(f_i)$, $\deg(h_i) = 0$, the root decomposition of $\hat{\mathfrak{g}}$ looks as follows:

$$\boxed{\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\substack{(\alpha,k)\neq 0\\ \alpha\in\Delta(\mathfrak{g})\cup\{0\},\, k\in\mathbb{Z}}} \mathfrak{g}_\alpha \cdot t^k} \qquad \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}\cdot K$$

(b) The root system $\Delta(\hat{\mathfrak{g}}) \subset \hat{Q}$ is expressed via the root system $\Delta(\mathfrak{g})$ as follows:

$$\boxed{\Delta(\hat{\mathfrak{g}}) = \Delta(\mathfrak{g}) \amalg \coprod_{k\in\mathbb{Z}\setminus\{0\}} \{\alpha + k\underbrace{(\alpha_0+\theta)}_{=:\delta} \mid \alpha\in\Delta(\mathfrak{g})\cup\{0\}\}}$$

(c) The set of positive roots $\Delta(\hat{\mathfrak{g}})_+ \subset \hat{Q}$ is as follows:

$$\boxed{\Delta(\hat{\mathfrak{g}})_+ = \Delta(\mathfrak{g})_+ \amalg \coprod_{k\in\mathbb{Z}_{>0}} \{\alpha + k\delta \mid \alpha\in\Delta(\mathfrak{g})\cup\{0\}\}}$$

Let $F := Q\otimes_{\mathbb{Z}}\mathbb{C}$ - the $\mathbb{C}$-vector space with the basis $\{\alpha_1,\ldots,\alpha_n\}$.

Def 2: Define the linear operator $F \to \mathfrak{h}^*$, $\alpha \mapsto \bar{\alpha}$, via $\bar{\alpha}_j(h_i) = a_{ij}$ ($i,j\in\{1,\ldots,n\}$)

Rmk 3: (a) $[h,x] = \bar{\alpha}(h)\cdot x$ $\forall h\in\mathfrak{h}$, $x\in\mathfrak{g}_\alpha$ ($\alpha\in\Delta$)

(b) The above map $F\to\mathfrak{h}^*$ is an isomorphism iff A-nondegenerate.

(c) In the case of $\hat{\mathfrak{g}}$ ($\mathfrak{g}$-simple f.d.), $\mathrm{Ker}(F\to\mathfrak{h}^*)$ is 1-dim spanned by $\delta = \alpha_0 + \theta$.

* Today: Representation theory of $\mathfrak{g}(A)$.

Let us first start from the case when A – Cartan matrix, so that $\mathfrak{g}(A)$ – simple f.d.

• Rep. theory of simple f.d. $\mathfrak{g}(A)$

Def 3: The category $\mathcal{O}$ of modules over $\mathfrak{g}=\mathfrak{g}(A)$ is defined as follows:

Obj$(\mathcal{O})$ = $\mathfrak{g}$-modules M satisfying:

(1) M is $\mathfrak{h}$-diagonalizable, i.e. $M=\bigoplus_{\mu\in\mathfrak{h}^*} M[\mu]$, $M[\mu]:=\{v\in M \mid h(v)=\mu(h)\cdot v\ \forall h\in\mathfrak{h}\}$

(2) $\dim(M[\mu])<\infty\ \forall\mu$

(3) $\exists\lambda_1,\ldots,\lambda_m\in\mathfrak{h}^*$ s.t. $\mathrm{Supp}(M):=\{\mu\in\mathfrak{h}^* \mid M[\mu]\neq 0\}\subseteq D(\lambda_1)\cup\ldots\cup D(\lambda_m)$, where

$$D(\lambda):=\{\lambda-n_1\alpha_1-\ldots-n_r\alpha_r \mid n_1,\ldots,n_r\in\mathbb{Z}_{\geq 0}\}\subset\mathfrak{h}^*.$$

Mor$(\mathcal{O})$ = $\mathfrak{g}$-module morphisms (note: it is automatic that $M[\mu]\to N[\mu]$)

Rmk 4: Consider the principal $\mathbb{Z}$-grading on $\mathfrak{g}=\mathfrak{g}(A)$ with $\deg(e_i)=1=-\deg(f_i)$, $\deg(h_i)=0$ (arises via Q-grading via $Q\to\mathbb{Z}$, $\alpha_i\mapsto 1$)

Then the above category $\mathcal{O}$ is clearly a refinement of the old def-n of category $\mathcal{O}$ (see Lecture 4)

In particular, $\forall\lambda\in\mathfrak{h}^*\rightsquigarrow$ Verma modules $M_\lambda=M_\lambda^+$, their irreducible quot-s L_λ are el-s of $\mathcal{O}$.

Also any graded submodule of $M\in\mathcal{O}$ and a quotient by a graded submodule are also el-s of $\mathcal{O}$.

Def 4: For $M\in\mathcal{O}$, its formal character

$$\boxed{\mathrm{ch}(M):=\sum_{\mu\in\mathfrak{h}^*}\dim(M[\mu])e^\mu}$$

which is an element of the ring $R:=\{\sum_{\mu\in\mathfrak{h}^*}a_\mu e^\mu \mid \text{supported on finite union of } D(\lambda)\text{'s}\}$

By condition (3) of Def 3, this definition of $\mathrm{ch}(M)$ is well-defined!

Example 1: $\boxed{\mathrm{ch}\,M_\lambda \overset{\text{PBW}}{=\!=} \dfrac{e^\lambda}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})}}$. In particular, $\{\mathrm{ch}\,M_\lambda\}$ form a "topological basis" of R, i.e. any $f\in R$ may be uniquely written as a "bounded from above" sum $\sum b_\lambda \mathrm{ch}(M_\lambda)$

Example 2: $A=(2)\Rightarrow\mathfrak{g}(A)\simeq\mathfrak{sl}_2$. Then $\mathfrak{sl}_2$-weights $\simeq\mathbb{C}$ via $\omega_1\longmapsto 1$ ($\alpha\mapsto 2$). Then if we denote e^{ω_1} by x, we get:

$$\boxed{\mathrm{ch}(M_\lambda)=\frac{x^\lambda}{1-x^{-2}}}$$

If $\lambda\in\mathbb{Z}_{\geq 0}$, then L_λ has the formal character

$$\mathrm{ch}(L_\lambda)=\mathrm{ch}(M_\lambda)-\mathrm{ch}(M_{-\lambda-2})=\frac{x^\lambda-x^{-\lambda-2}}{1-x^{-2}}=\boxed{\frac{x^{\lambda+1}-x^{-\lambda-1}}{x-x^{-1}}}$$

Simplest example of Weyl-Kac formula to be discussed in the next class

$e^\mu\cdot e^\nu:=e^{\mu+\nu}$

Lemma 1: (a) If $M_1,M_2\in\mathcal{O}$, then $M_1\otimes M_2\in\mathcal{O}$ and $\boxed{\mathrm{ch}(M_1\otimes M_2)=\mathrm{ch}(M_1)\cdot\mathrm{ch}(M_2)}$

(b) If $0\to N\to M\to M/N\to 0$ is a short exact sequence in $\mathcal{O}$ $\Rightarrow$ $\boxed{\mathrm{ch}(M)=\mathrm{ch}(N)+\mathrm{ch}(M/N)}$

▸ (a) Follows immediately from $(M_1\otimes M_2)[\mu]=\bigoplus_{\mu_1+\mu_2=\mu}M_1[\mu_1]\otimes M_2[\mu_2]$

(b) Follows immediately from $0\to N[\mu]\to M[\mu]\to (M/N)[\mu]\to 0$

■

Exercise: Provide two modules $M_1,M_2\in\mathcal{O}$ such that $M_1\not\simeq M_2$, but $\mathrm{ch}(M_1)=\mathrm{ch}(M_2)$ (use Lemma 1(b))

$\boxed{Q}$: Can this be generalized to any Kac-Moody algebra?

The problem is that e.g. Verma modules are no longer in cat. $\mathcal{O}$ in general. Indeed, consider vectors $\{ht^k(v_\lambda) \mid h\in\mathfrak{h}, k<0\}$ in the Verma module M_λ over $\hat{\mathfrak{g}}$. All of them have weight λ!

To work around this problem, we will extend the Cartan subalgebra:

Def 5: Let $A\in \mathrm{Mat}_{r\times r}(\mathbb{C})$ and $\mathfrak{g}(A)$ be the corresponding contragredient Lie algebra. Define $\boxed{\mathfrak{g}_{ext}(A) := \mathfrak{g}(A)\oplus \mathbb{C}D_1\oplus\mathbb{C}D_2\oplus\ldots\oplus\mathbb{C}D_r}$, where

$$[D_i,D_j]=0,\ [D_i,e_i]=e_i,\ [D_i,f_i]=-f_i,\ [D_i,h_i]=0,\ [D_i,(e,h,f)_j]=0 \text{ for } j\neq i$$

Alternatively, $\mathfrak{g}_{ext}(A) = \underset{\uparrow \text{basis } \{D_1,\ldots,D_r\}}{\mathbb{C}^r} \ltimes \mathfrak{g}(A)$. In particular, we have:

$$\boxed{\mathfrak{g}_{ext}(A)\simeq \mathfrak{n}_-\oplus\mathfrak{h}_{ext}\oplus\mathfrak{n}_+,\quad \mathfrak{h}_{ext}=\mathfrak{h}\oplus\mathbb{C}D_1\oplus\ldots\oplus\mathbb{C}D_r}$$

Note: $\dim\mathfrak{h}_{ext}=2r=2\dim\mathfrak{h}$.

Recall that (right before Def 2) we defined $\bar{\alpha}_j\in\mathfrak{h}^*$ via $\bar{\alpha}_j(h_i)=a_{ij}$, which allowed to view every $\alpha\in Q$ as a functional on $\mathfrak{h}$. The corresponding linear map $F:=Q_\mathbb{C}\longrightarrow\mathfrak{h}^*$ is not isom. unless A-nondegen.

BUT NOW: We will view each α_j as a functional $\mathfrak{h}_{ext}\to\mathbb{C}$ via $\boxed{h_i\overset{\alpha_j}{\mapsto}a_{ij},\ D_i\overset{\alpha_j}{\mapsto}\delta_{ij}}$ $\rightsquigarrow$ $\boxed{Q_\mathbb{C}\to\mathfrak{h}^*_{ext}}$

Note: $[h,x]=\alpha(h)\cdot x\quad\forall h\in\mathfrak{h}_{ext},\ x\in\mathfrak{g}_\alpha\ (\alpha\in\Delta)$.

Set as before $F:=Q\otimes_\mathbb{Z}\mathbb{C}=\mathbb{C}\alpha_1\oplus\ldots\oplus\mathbb{C}\alpha_r$ and

$$\boxed{P:=\mathfrak{h}^*\oplus F=\mathbb{C}h_1^*\oplus\ldots\oplus\mathbb{C}h_r^*\oplus\mathbb{C}\alpha_1\oplus\ldots\oplus\mathbb{C}\alpha_r}$$

Here h_j^* is viewed as a functional $\mathfrak{h}_{ext}\to\mathbb{C}$ via $\boxed{h_i\mapsto\delta_{ij},\ D_i\mapsto 0}$

So: We have a natural linear map $\varphi: P\longrightarrow\mathfrak{h}^*_{ext}$ and it is an isomorphism!

this follows by noticing that the matrix consisting of h_i^*, α_i evaluated at h_j, D_j is $\left(\begin{array}{c|c} I & * \\ \hline 0 & I\end{array}\right)\Rightarrow$ nondegenerate.

! After this modification ($\mathfrak{g}(A)\rightsquigarrow\mathfrak{g}_{ext}(A)$, $\mathfrak{h}^*\rightsquigarrow\mathfrak{h}^*_{ext}$) we may define all the above notions we had for simple $\mathfrak{g}=\mathfrak{g}(A)$:

$\boxed{\text{Category } \mathcal{O},\ \text{Verma modules } M_\lambda\ (\lambda\in P),\ \text{irreducibles } L_\lambda\ (\lambda\in P),\ \mathrm{ch}\, M}$

Rmk 5: In Feigin-Zelevinsky, P is the same, but they do not extend $\mathfrak{g}(A)$ to $\mathfrak{g}_{ext}(A)$. As a result, their definition of category $\mathcal{O}$ over $\mathfrak{g}(A)$ is not intrinsic, but requires an extra P-grading: $\begin{cases} M=\bigoplus_{\mu\in P} M[\mu],\ \dim(M[\mu])<\infty \\ \mathrm{supp}(M)\subseteq D(\lambda_1)\cup\ldots\cup D(\lambda_m)\end{cases}$ AND $\begin{cases}\mathfrak{g}_\alpha M[\mu]\subseteq M[\mu+\alpha] \\ hv=\mu(h)v\quad\forall h\in\mathfrak{h}, v\in M[\mu]\end{cases}$

Here: $D(\lambda)\subseteq P$ is defined as $\{\lambda-n_1\alpha_1-\ldots-n_r\alpha_r\in P \mid n_1,\ldots,n_r\in\mathbb{Z}_{\geq 0}\}$ for any $\lambda\in P$.

Also: Their $\mathrm{Mor}(\mathcal{O})$ are P-graded $\mathfrak{g}$-module morphisms.

Lemma 2: For $\lambda\in P$: $\boxed{\mathrm{ch}(M_\lambda)=e^\lambda\cdot\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{-\dim(\mathfrak{g}_{-\alpha})=-\dim\mathfrak{g}_\alpha}}$

▸ Follows as before from PBW ▪

Let us note right away that even for simple f.dim. $\mathfrak{g}=\mathfrak{g}(A)$ this notion of category $\mathcal{O}$ differs from the one we started from. However, they are equivalent as explained below.

<u>Lemma 3</u>: For $\chi\in\mathfrak{h}^*$, let $\mathcal{O}_\chi$ denote the full subcategory of $\mathcal{O}$ with weights in $\chi+F\subseteq P$.

(a) The category $\mathcal{O}$ naturally decomposes into the direct sum $\bigoplus_{\chi\in\mathfrak{h}^*}\mathcal{O}_\chi$

(b) If $\chi_1,\chi_2\in\mathfrak{h}^*$ and $\chi_1-\chi_2=\bar{\alpha}$ for some $\alpha\in F$ (as before $\bar{\alpha}$ denotes corresp. elt of $\mathfrak{h}^*$), then categories $\mathcal{O}_{\chi_1}$ and $\mathcal{O}_{\chi_2}$ are naturally isomorphic.

(c) If A is nonsingular, then every $\mathcal{O}_\chi$ is naturally isomorphic to $\mathcal{O}_0$.

for $\mathfrak{g}=\mathfrak{g}(A)$ - simple f.d. $\mathcal{O}_0$ coincides with category $\mathcal{O}$ which we defined first.

► (a) Clear: For $M\in\mathcal{O}$, set $M(\chi):=\bigoplus_{\mu\in\chi+F}M[\mu]$. Then $M=\bigoplus_{\chi\in\mathfrak{h}^*}M(\chi)$ and each $M(\chi)\in\mathcal{O}_\chi$. Moreover, $\mathrm{Hom}_{\mathcal{O}}(M,M')=0$ if $M\in\mathcal{O}_\chi$, $M'\in\mathcal{O}_{\chi'}$, $\chi\neq\chi'$.

(b) Let $M\in\mathcal{O}_{\chi_2}$. Denote by M' the module from $\mathcal{O}$ which coincides with M as a $\mathfrak{g}(A)$-module (but <u>not</u> as $\mathfrak{g}^{ext}(A)$-module) and $\boxed{M'_\chi:=M_{\chi-\chi_1+\chi_2+\alpha}}$. Obviously $M'\in\mathcal{O}_{\chi_1}$. Moreover, the function $M\mapsto M'$ establishes the isom. of categories $\mathcal{O}_{\chi_2}\xrightarrow{\sim}\mathcal{O}_{\chi_1}$. ($M\mapsto M'$ changes the action of D_i by common constants)

(c) Follows from (b), since $F\xrightarrow{\sim}\mathfrak{h}^*$ - isom. if A - nondeg. ∎

<u>Rmk 6</u>: (a) If $\mathfrak{g}(A)$ - simple, then all $\mathcal{O}_\chi$ are the same (as usual category $\mathcal{O}$ for $\mathfrak{g}(A)$).

(b) If $\mathfrak{g}(A)$ - affine KM, then $F\to\mathfrak{h}^*$ has 1-dim kernel $\Rightarrow$ image has codim $=1$. Hence, there is essentially 1-parameter family $\mathcal{O}(k)$, $k\in\mathbb{C}$, of categories in $\mathcal{O}$. In particular, if $\mathfrak{g}(A)=\hat{\mathfrak{g}}$, then this k is the <u>level</u>, i.e. the value of functional on K.

(c) Let us also note that while for $\mathfrak{g}=\mathfrak{g}(A)$ - simple f.d., its adjoint repr-n is L_θ and is in category $\mathcal{O}$, the adjoint repr-n of general $\mathfrak{g}(A)$ doesn't belong to $\mathcal{O}$.

<u>Lemma 4</u>: (a) The center Z of $\mathfrak{g}(A)$ is $\{\sum\beta_i h_i\mid\beta_i\in\mathbb{C},\ \sum\beta_i a_{ij}=0\ \forall j\}$. So $\dim(Z)=\dim(\mathrm{Ker}\,A)$

(b) If A is a generalized Cartan matrix, then $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$

(c) If A is an indecomposable symmetrizable matrix, then any proper graded ideal of $\mathfrak{g}$ is contained in Z. In particular, if A - nondeg., then $\mathfrak{g}(A)$ has no proper graded ideals.

► (a) If $x\in Z$, then each homogeneous component of x is central $\Rightarrow$ may assume $x\in\mathfrak{g}_\alpha$. If $\alpha\neq0$, then $\mathbb{C}x$ is a graded ideal non-intersecting $\mathfrak{h}\Rightarrow$ ↯. Thus, $\alpha=0$, i.e. $x\in\mathfrak{h}\Rightarrow x=\sum\beta_i h_i$. But then x is central iff $[x,e_j]=0=[x,f_j]\ \forall j\Leftrightarrow\sum\beta_i a_{ij}=0\ \forall j$.

(b) It suffices to request $a_{ii}\neq0\ \forall i$. Then $\{e_i,h_i,f_i\}\subset[\mathfrak{g},\mathfrak{g}]\Rightarrow[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ as $\mathfrak{g}$ is gen-d by $\{e_i,h_i,f_i\}$

(c) If $0\neq I\neq\mathfrak{g}(A)$ is a graded ideal, then $I=I_+\oplus I_0\oplus I_-$, $I_\pm:=I\cap\mathfrak{n}_\pm$, $I_0:=I\cap\mathfrak{h}\neq0$. Suppose $I\not\subseteq Z$. Then I_+ or I_- are nonzero! (if $I_+=I_-=0\Rightarrow\exists h\in I\setminus Z\Rightarrow[h,e_i]\in I_+$ and it is a nonzero multiple of e_i for some i) WLOG assume $I_+\neq0$. Pick a homogeneous nonzero element $a\in(I_+)_\alpha$. Let J be the ideal of $\mathfrak{g}(A)$ generated by a, so that $0\neq J\subseteq I$ and $J\cap\mathfrak{h}\neq0$. The latter implies that there exist $i_1,\dots,i_n,j_1,\dots,j_m$ such that $x:=f_{i_1}\dots f_{i_n}e_{j_1}\dots e_{j_m}a\in\mathfrak{h}\setminus\{0\}$.
<u>But</u>: then x is a nonzero multiple of $h_{i_1}\Rightarrow\langle e_{i_1},h_{i_1},f_{i_1}\rangle\in\mathfrak{g}(A)$.
If $a_{i_1j}\neq0\Rightarrow\langle e_j,h_j,f_j\rangle\in J$ etc... Using indecomposability of $A\Rightarrow$ all $e_j,h_j,f_j\in J\Rightarrow$ ↯ ∎

• Invariant Form

Let A be an indecomposable complex matrix. We want to classify symmetric forms

$$\boxed{\begin{array}{lll} (\cdot,\cdot): \mathfrak{g}(A)\times\mathfrak{g}(A)\longrightarrow\mathbb{C} & \text{such that} & 1)\ (\mathfrak{g}_\alpha,\mathfrak{g}_\beta)=0 \text{ if } \alpha+\beta\neq 0 \\ (\cdot,\cdot): \tilde{\mathfrak{g}}(A)\times\tilde{\mathfrak{g}}(A)\longrightarrow\mathbb{C} & & 2)\ (\cdot,\cdot)\text{-invariant}\end{array}} \longleftarrow (\,,\,) \text{ is of degree ZERO}$$

Note: We do not require $(\cdot,\cdot)$ to be nondegenerate!

Set $d_i := (e_i, f_i)$ $\forall i$

Assumption: $d_i \neq 0$ $\forall i$ (if $d_i = 0$ then the form $(\cdot,\cdot)$ is too degenerate to be interesting)

Note:

$$\begin{array}{l}(h_i,h_j)=(h_i,[e_j,f_j])\overset{\text{invariance}}{=\!=\!=}([h_i,e_j],f_j)=a_{ij}\cdot d_j \\ \quad\,\shortparallel \\ (h_j,h_i)=(h_j,[e_i,f_i])\overset{\text{invariance}}{=\!=\!=}([h_j,e_i],f_i)=a_{ji}\cdot d_i\end{array} \implies \boxed{a_{ij}d_j=a_{ji}d_i}$$

So: For such $(\cdot,\cdot)$ to exist, we need to require that A is symmetrizable: $\boxed{(AD)^T=AD}$,
where D is diagonal & nondegenerate (if $\mathfrak{g}(A)$-Kac-Moody, may choose D to have el-s of $\mathbb{Q}_{>0}$ on diagonal)

Exercise: If A is indecomposable symmetrizable, then $\underset{\text{diagonal}}{D}$ s.t. $(AD)^T=AD$ is unique up to a scaling.

Rmk 7: In our previous discussions, we required $(DA)^T=DA$ but $(DA)^T=DA \Leftrightarrow (AD^{-1})^T=AD^{-1}$

From now on, let us assume that A is symmetrizable!

Lemma 5: If A is indecomposable symmetrizable, then there is at most one (up to scaling) symmetric invariant form $(\,,\,):\tilde{\mathfrak{g}}(A)\times\tilde{\mathfrak{g}}(A)\to\mathbb{C}$ (resp. $\mathfrak{g}(A)\times\mathfrak{g}(A)\to\mathbb{C}$) of degree ZERO.

▶ Let $\mathfrak{g}$ stay for $\tilde{\mathfrak{g}}(A)$ or $\mathfrak{g}(A)$ resp. Then $(\,,\,):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ may be viewed as a linear map $\mathfrak{g}\xrightarrow{\gamma}\mathfrak{g}^*$ (here $\mathfrak{g}^*$ denotes the restricted dual of Q-graded $\mathfrak{g}$). Moreover, $(\cdot,\cdot)$-invariant iff γ-$\mathfrak{g}$-module homomorphism. As $\mathfrak{g}$ is generated by $\{e_i,h_i,f_i\}$, and actually $\{e_i,f_i\}_{i=1}^n$, it suffices to show that $\{\gamma(e_i),\gamma(f_i)\}_{i=1}^n$ are unique up to a common scalar.
But by above discussion: $\gamma(e_i)=d_if_i^*$, $\gamma(f_i)=d_ie_i^*$, where $D=\mathrm{diag}(d_1,\dots,d_n)$ symmetrizes A, and by above exercise such D is unique up to a scalar ∎

Theorem 1: If A is an indecomposable symmetrizable, there exists a nonzero symmetric invariant form of degree ZERO on $\tilde{\mathfrak{g}}(A)$ and $\mathfrak{g}(A)$.

▶ • It suffices to treat the case of $\mathfrak{g}(A)$, since having constructed such $(\,,\,):\mathfrak{g}(A)\times\mathfrak{g}(A)\to\mathbb{C}$ its composition with the natural projection $\tilde{\mathfrak{g}}(A)\times\tilde{\mathfrak{g}}(A)\twoheadrightarrow\mathfrak{g}(A)\times\mathfrak{g}(A)$ gives rise to the claimed form on $\tilde{\mathfrak{g}}(A)$. Note, in particular, that: $I=\mathrm{Ker}(\tilde{\mathfrak{g}}(A)\twoheadrightarrow\mathfrak{g}(A))\subseteq\mathrm{Ker}(\cdot,\cdot)_{\tilde{\mathfrak{g}}(A)}$.

• Note that $\boxed{\mathrm{Ker}(\cdot,\cdot)_{\mathfrak{g}(A)}=Z}$ - the center of $\mathfrak{g}(A)$. Indeed, $J:=\mathrm{Ker}(\cdot,\cdot)_{\mathfrak{g}(A)}$ is a graded ideal, $J\neq\mathfrak{g}(A)$ as $(e_i,f_i)\neq 0 \Rightarrow e_i\notin J$. Hence, $J\subseteq Z$ by Lemma 4(c). But, we also have $Z\subseteq J$: $x=\sum \beta_ih_i\in Z \Leftrightarrow \sum\beta_ia_{ij}=0$ $\forall j$ $\Leftrightarrow (x,h_j)=0$ $\forall j$ $\Leftrightarrow x\in\mathrm{Ker}(\cdot,\cdot)_{\mathfrak{g}(A)}$ as $(\cdot,\cdot)$ is of degree ZERO.

It remains to construct such $(\cdot,\cdot):\mathfrak{g}(A)\times\mathfrak{g}(A)\to\mathbb{C}$.

(Continuation of proof of Theorem 1)

• Let us now construct the claimed $(\, , \,): \mathfrak{g}(A) \times \mathfrak{g}(A) \longrightarrow \mathbb{C}$. We shall follow [Feigin – Zelevinsky, pp. 51-52]

For $k \geq 1$, set $\boxed{\mathfrak{g}^k := \bigoplus_{\substack{\alpha \in \Delta \cup \{0\} \\ |\alpha| \leq k}} \mathfrak{g}_\alpha}$, where $|\sum k_i \alpha_i| := \sum |k_i|$

We will construct $(\cdot, \cdot): \mathfrak{g}^k \times \mathfrak{g}^k \longrightarrow \mathbb{C}$ inductively in k.

Base of Induction: $k=1$

$\mathfrak{g}^1 = \mathfrak{h} \oplus \bigoplus_{i=1}^{n} \mathbb{C} e_i \oplus \bigoplus_{i=1}^{n} \mathbb{C} f_i$. Set $(e_i, f_i) = d_i$, $(h_i, h_j) = d_j a_{ij} = d_i a_{ji}$

$(e_i, e_j) = (f_i, f_j) = (e_i, h_j) = (f_i, h_j) = (e_i, f_{i'}) = 0 \quad \forall i, j, \; i' \neq i.$

Then: $([x,y], z) = (x, [y,z])$ for any $x, y, z \in \mathfrak{g}^1$ s.t. $[x,y], [y,z] \in \mathfrak{g}^1$, which essentially boils to $\underset{a_{ij} \cdot d_j \cdot \delta_{jj'}}{\underbrace{([h_i, e_j], f_{j'})}} = \underset{\delta_{jj'} \cdot d_j a_{ij}}{\underbrace{(h_i, [e_j, f_{j'}])}}$, $\underset{-a_{ij} \cdot d_j \cdot \delta_{jj'}}{\underbrace{([h_i, f_j], e_{j'})}} = \underset{\delta_{jj'} \cdot (-d_j a_{ij})}{\underbrace{(h_i, [f_j, e_{j'}])}}$

Step of Induction: Constructed $(\cdot, \cdot): \mathfrak{g}^k \times \mathfrak{g}^k \longrightarrow \mathbb{C}$ and need to extend to $\mathfrak{g}^{k+1} \times \mathfrak{g}^{k+1} \longrightarrow \mathbb{C}$.

Let $\alpha \in \Delta_+$ s.t. $|\alpha| = k+1$, and let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. Then x may be written as $x = \sum_k [a_k, b_k]$, where $a_k \in \mathfrak{g}^1 \subseteq \mathfrak{g}^k$, $b_k \in \mathfrak{g}^k$. Set:

$$\boxed{(x, y) = (y, x) := \sum_k (a_k, [b_k, y])}$$

∘ First of all, we need to verify this is well-defined, i.e. $\sum_k [a_k, b_k] = 0 \Longrightarrow \sum (a_k, [b_k, y]) = 0$.

It suffices to consider $y = [u, v]$, $u, v \in \mathfrak{g}^k$. Then:

$$(a_k, [b_k, y]) = (a_k, [b_k, [u,v]]) \overset{\text{Jacobi}}{=\!=} (a_k, [[b_k, u], v]) + (a_k, [u, [b_k, v]])$$

$$\overset{\text{Induction Assumption}}{=\!=} ([a_k, [b_k, u]], v) + ([[b_k, v], a_k], u) \overset{\text{Induct. Assum}}{=\!=} (v, [a_k, [b_k, u]]) + ([b_k, v], [a_k, u])$$

$$\overset{-''-}{=\!=} (v, [a_k, [b_k, u]]) + (v, [[a_k, u], b_k]) \overset{\text{Jacobi}}{=\!=} (v, [[a_k, b_k], u])$$

$$\Downarrow$$

$$\sum_k (a_k, [b_k, \underbrace{[u,v]}_{y}]) = (v, [\underbrace{\sum [a_k, b_k]}_{=0}, u]) = 0. \quad \checkmark$$

∘ It remains to show that $(\cdot, \cdot)$ is invariant on $\mathfrak{g}^{k+1}$. Let $\alpha \in \Delta_+$ with $|\alpha| = k+1$.

Then, we just need to prove the following two equalities:

1) $([x,y], z) = (x, [y,z])$ for $x \in \mathfrak{g}_{\alpha-\beta}$, $y \in \mathfrak{g}_\beta$, $z \in \mathfrak{g}_{-\alpha}$ and $\beta \in \Delta_+$ with $|\beta| \leq k$.

▷ $\alpha - \beta \in \Delta_+$ (otherwise, both sides are ZERO), hence, the equality follows from our definition □

2) $([x,y], z) = (x, [y,z])$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\beta}$, $z \in \mathfrak{g}_{\beta-\alpha}$ and $\beta \in \Delta_+$ with $|\beta| \leq k+1$.

▷ WLOG may assume $x = [a, b]$, $a, b \in \mathfrak{g}^k$. Then:

$$([[a,b], y], z) \overset{\text{Jacobi}}{=} ([a, [b, y]], z) + ([[a, y], b], z) \overset{\text{Induction}}{=\!=} (a, [[b, y], z]) + (b, [z, [a, y]])$$

$$\overset{\text{Induction}}{=} (a, [[b, y], z]) + ([b, z], [a, y]) \overset{\text{Induction}}{=\!=} (a, [[b, y], z]) + (a, [y, [b, z]]) \overset{\text{Jacobi}}{=\!=} (a, [b, [y, z]]) \overset{\text{Ind}}{=} ([a,b], [y,z]) \; \square$$

This completes our proof of Theorem 1. ■

Recall that we started from $F := \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{C} \to \mathfrak{h}^*$ sending $\alpha_i \mapsto \bar{\alpha}_i$ s.t. $\bar{\alpha}_i(h_j) = a_{ji}$.

Define now $\boxed{\gamma : F \to \mathfrak{h} \text{ via } \alpha_i \mapsto d_i^{-1} h_i =: h_{\alpha_i}}$ This is clearly a vector space isomorphism.

Def 6: For $\alpha \in F$, define $h_\alpha \in \mathfrak{h}$ via $h_\alpha := \gamma(\alpha)$

Lemma 6: $\forall \alpha \in F, h \in \mathfrak{h}$: $\boxed{(h_\alpha, h) = \bar{\alpha}(h)}$

▸ $(h_{\alpha_i}, h_j) = (d_i^{-1} h_i, h_j) = d_i^{-1}\cdot(d_i a_{ji}) = a_{ji} = \bar{\alpha}_i(h_j)$ ∎

Lemma 7: If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x,y] = (x,y)\cdot h_\alpha$.

▸ May assume WLOG that $\alpha \in \Delta_+$. We shall prove by induction in $|\alpha|$.

- $|\alpha| = 1 \Rightarrow \alpha = \alpha_i \Rightarrow$ just need to verify $[e_i, f_i] = \underbrace{(e_i, f_i)}_{d_i}\underbrace{h_{\alpha_i}}_{d_i^{-1}h_i} = h_i$. ✓
- $|\alpha| = k+1$ $(k>0)$: it suffices to treat $x = [a,b]$ with $a \in \mathfrak{g}_\beta$, $b \in \mathfrak{g}_\gamma$ with $\beta+\gamma = \alpha$, $\beta,\gamma \in \Delta^+$, $|\beta|, |\gamma| \leq k$

Then: $[x,y] = [[a,b],y] \overset{\text{Jacobi}}{=} [[a,y],b] + [a,[b,y]] \overset{\text{Induction Assumption}}{=} (a,[b,y])h_\beta - (b,[a,y])h_\gamma$

$\overset{\text{Invariance}}{=} ([a,b],y)h_\beta + ([a,b],y)h_\gamma = (x,y)\cdot(h_\beta + h_\gamma) = (x,y)h_\alpha$ ✓ ∎

Let us now endow $\mathfrak{g}(A)$ with the principal $\mathbb{Z}$-grading, i.e. $\deg(e_i) = 1$, $\deg(h_i) = 0$, $\deg(f_i) = -1$.
Then $\mathfrak{g}(A)[0] = \mathfrak{h}$; $\mathfrak{g}(A)[\pm n] = \bigoplus_{\alpha\in\Delta_\pm : |\alpha| = n} \mathfrak{g}(A)[\alpha]$.

Lemma 8: $\mathfrak{g}(A)$ is a nondegenerate $\mathbb{Z}$-graded Lie algebra (in the sense of Lecture 3).

▸ As noted in the proof of Thm 1: $\mathrm{Ker}(\cdot,\cdot)_{\mathfrak{g}(A)} = \mathfrak{Z} \underset{\text{Lemma 4(a)}}{\subseteq} \mathfrak{h}$, hence, $\forall \alpha \in \Delta$: $(\cdot,\cdot) : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is nondeg.
Due to Lemma 7: $\lambda([x,y]) = (x,y)\cdot\lambda(h_\alpha)$. Hence if $\lambda \in \mathfrak{h}^*$ is such that $\lambda(h_\alpha) \neq 0$ $\forall$ roots α of $\mathfrak{g}(A)$, then $\lambda([x,y]) : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is nondeg. ∎

Let us conclude by introducing the inner product $(\cdot,\cdot)$ on $P := \mathfrak{h}^* \oplus F = \mathfrak{h}^*_{ext}$

$$\boxed{(\cdot,\cdot) : P\times P \to \mathbb{C} \quad \text{via} \quad (\varphi+\alpha, \psi+\beta) = \varphi(h_\beta) + \psi(h_\alpha) + (h_\alpha, h_\beta) \quad \text{for } \varphi,\psi \in \mathfrak{h}^*,\ \alpha,\beta \in F}$$

It is obvious that $(\cdot,\cdot)$ is symmetric.
Also it is non-degenerate! Indeed, in the basis $\{h^*_{\alpha_i}, \alpha_i\}$ of P this pairing is given by $\left(\begin{array}{c|c} 0 & I \\ \hline I & * \end{array}\right)$ $D^{-1}A$
But: recall that $P \xrightarrow{\sim} \mathfrak{h}^*_{ext}$.
Thus, above nondegenerate form on P gives rise to a nondegenerate pairing on $\mathfrak{h}_{ext}$.

Exercise: Verify that $(D_i, D_j) = 0$, $(D_i, h_{\alpha_j}) = \delta_{ij}$, $(h_{\alpha_i}, h_{\alpha_j}) = d_i^{-1} a_{ij}$

Cor 1: The extension of $(\,,\,)$ on $\mathfrak{g}(A)$ to $\mathfrak{g}_{ext}(A)$ is a nondegenerate symm. invariant form.