* Last time:
- Category $\mathcal{O}$ and formal characters for $\mathfrak{g}=\mathfrak{g}(A)$-simple f.dim.
- Generalization to any $\mathfrak{g}(A)$ by extending $\mathfrak{g}(A) \rightsquigarrow \mathfrak{g}_{ext}(A)=\underset{\langle D_1,\ldots,D_r\rangle}{\mathbb{C}^r}\ltimes\mathfrak{g}(A)$, $\mathfrak{h}\rightsquigarrow\mathfrak{h}_{ext}=\mathfrak{h}\oplus\overset{\mathbb{C}D_1\oplus\ldots\oplus\mathbb{C}D_r}{\overset{\|}{\mathbb{C}^r}}$
- A-indecomposable, symmetric $\rightsquigarrow\exists$ unique (up to a scalar) invariant degree ZERO pairing $(\cdot,\cdot):\mathfrak{g}(A)\times\mathfrak{g}(A)\longrightarrow\mathbb{C}$, $\tilde{\mathfrak{g}}(A)\times\tilde{\mathfrak{g}}(A)\longrightarrow\mathbb{C}$
- $\gamma: F:=Q\otimes_{\mathbb{Z}}\mathbb{C}\xrightarrow{\sim}\mathfrak{h}$, $\alpha_i\mapsto d_i^{-1}h_i$ s.t. $(\underset{\gamma(\alpha)}{h_\alpha},h)=\alpha(h)$.
- Nondegenerate inner product on $P:=\mathfrak{h}^*\oplus F$ $\Downarrow$ Nondegenerate symm. invariant pairing on $\mathfrak{g}_{ext}(A)$.

* Casimir operator for $\mathfrak{g}(A)$

|| **Def 1**: Define $\rho\in\underset{\subset P}{\mathfrak{h}^*}$ via $\boxed{\rho(h_i)=\frac{a_{ii}}{2}}$ $\forall i$

> Recall $(\,,\,):P\times P\longrightarrow\mathbb{C}$ is nondegenerate, $P \twoheadrightarrow \mathfrak{h}^*_{ext}$; hence $\exists!\ h_\rho\in\mathfrak{h}_{ext}$ s.t. $(\rho,\mu)=\mu(h_\rho)\ \forall\mu\in\mathfrak{h}^*_{ext}$

Note: For Kac–Moody algebras $a_{ii}=2\Rightarrow\boxed{\rho(h_i)=1.}$

Recalling the pairing $(\cdot,\cdot)$ on P from the end of Lecture 16, we get:

$$\boxed{(\rho,\rho)=0,\quad(\rho,\alpha_i)=\frac{a_{ii}}{2d_i}=\frac{1}{2}(\alpha_i,\alpha_i)}$$

We would like to generalize the Casimir element for a simple f.d. $\mathfrak{g}=\mathfrak{g}(A)$ to any contragredient $\mathfrak{g}(A)$.

$$\boxed{C=\sum_{\substack{a\in B\text{-orthonormal}\\ \text{basis of }\mathfrak{g}\\ \text{w.r.t. standard form}}}a^2=\sum_{\substack{x_i\text{-orthonormal}\\ \text{basis of }\mathfrak{h}}}x_i^2+\sum_{\alpha>0}\underset{\text{renormalized via }(e_\alpha,f_\alpha)=1}{(f_\alpha e_\alpha+e_\alpha f_\alpha)}=\sum_{i=1}^{r}x_i^2+h_{2\rho}+2\sum_{\alpha>0}f_\alpha e_\alpha}$$

But for general $\mathfrak{g}(A)$ this does not make sense as the sum $\sum_{\alpha>0}$ may be infinite!

Instead: We shall obtain a Casimir operator acting on any module $M\in\mathcal{O}$ and commuting with $\mathfrak{g}(A)$-action!

For any $\alpha\in\Delta_+$, let $\{e_\alpha^{(i)}\}_i$ and $\{f_\alpha^{(i)}\}_i$ be dual bases of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, respectively:

$\boxed{(e_\alpha^{(i)},f_\alpha^{(j)})=\delta_{ij}}$ $\longleftarrow$ Recall: $\forall\alpha\in\Delta$, the pairing $(\,,\,):\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}\longrightarrow\mathbb{C}$ is nondegenerate.

Define

$$\boxed{\begin{array}{l}\Delta_+:=2\sum_{\alpha>0}\sum_{i=1}^{\dim\mathfrak{g}_\alpha}f_\alpha^{(i)}e_\alpha^{(i)}\ \text{– well-defined operator } M\circlearrowleft \text{ for any } M\in\mathcal{O}\\ \Delta_0:=\sum_{\substack{x_i\text{-orthonormal}\\ \text{basis of }\mathfrak{h}_{ext}}}x_i^2+\underset{=2h_\rho\leftarrow\text{see above}}{h_{2\rho}}:M\circlearrowleft\ \text{ or more explicitly }\ \Delta_0|_{M[\mu]}=(\mu,\mu+2\rho)\ \forall\mu\in\mathfrak{h}^*_{ext}\end{array}}$$

|| **Def 2**: For any $M\in\mathcal{O}$, the Casimir operator $\Delta:M\to M$ is defined via $\boxed{\Delta=\Delta_++\Delta_0}$

Theorem 1: (a) The operator Δ commutes with the $\mathfrak{g}(A)$-action.

(b) Δ acts on $\underset{\uparrow\ \text{Verma module}}{M_\lambda}$ by $(\lambda,\lambda+2\rho)\cdot\mathrm{Id}_{M_\lambda}$.

Rmk 1: (a) If M is a $\mathfrak{g}(A)$-module of highest weight $\lambda\in P$, then Δ acts on M via $(\lambda,\lambda+2\rho)\cdot\mathrm{Id}_M$.

(b) To see $\mu(h_\alpha)=(\alpha,\mu)$ $\forall\alpha\in F\subseteq P$, $\mu\in P$, it suffices to treat $\alpha=\alpha_i$ and $\mu\in\mathfrak{h}^*$ or $\mu=\alpha_j$.

If $\mu\in\mathfrak{h}^*\Rightarrow(\mu,\alpha)=\mu(h_\alpha)$ by definition. If $\mu=\alpha_j$, then $(\alpha_j,\alpha_i)=(h_{\alpha_j},h_{\alpha_i})=d_i^{-1}a_{ij}=d_i^{-1}\alpha_j(h_i)=\alpha_j(h_{\alpha_i})$

Proof of Theorem 1

(b) Let v_λ be the highest weight vector of M_λ. Then: $\Delta_+(v_\lambda)=0$, $\Delta_0(v_\lambda)=(\lambda,\lambda+2\rho)\cdot v_\lambda \Rightarrow$
$\Rightarrow \Delta(v_\lambda)=(\lambda,\lambda+2\rho)\cdot v_\lambda$. But M_λ is $\mathfrak{g}(A)$-generated by v_λ and Δ commutes with $\mathfrak{g}(A)$-action by (a)
$\Rightarrow \Delta_{|M_\lambda}=(\lambda,\lambda+2\rho)\cdot Id_{M_\lambda}$.

(a) It suffices to verify that Δ commutes with $\{e_k, f_k\}$ (as those el-s generate $\mathfrak{g}(A)$).

- For $v\in M[\mu]$, we have:

$$[\Delta_0,e_k](v)=\{(\mu+\alpha_k,\mu+2\rho+\alpha_k)-(\mu,\mu+2\rho)\}\cdot e_k(v)=\{(2\mu+2\rho,\alpha_k)+(\alpha_k,\alpha_k)\}\cdot e_k(v)$$
$$\overset{(2\rho,\alpha_k)=(\alpha_k,\alpha_k)}{=\!=} 2(\mu+\alpha_k,\alpha_k)\cdot \underbrace{e_k(v)}_{\in M[\mu+\alpha_k]} = 2h_{\alpha_k}e_k(v).$$

So: $\boxed{[\Delta_0,e_k]=2h_{\alpha_k}e_k}$ (as endomorphisms of M).

- Let us now prove $\boxed{[\Delta_+,e_k]=-2h_{\alpha_k}e_k}$

$$[\Delta_+,e_k]=2\sum_{\alpha>0}^{i}[f_\alpha^{(i)}e_\alpha^{(i)},e_k]=2\sum_{\alpha>0}^{i}(f_\alpha^{(i)}[e_\alpha^{(i)},e_k]-[e_k,f_\alpha^{(i)}]e_\alpha^{(i)}) \;=$$
$$= -2h_{\alpha_k}e_k+2\underbrace{\left(\sum_{\alpha>0}^{i}f_\alpha^{(i)}[e_\alpha^{(i)},e_k]-\sum_{\substack{\alpha>0\\ \alpha\neq\alpha_k}}^{i}[e_k,f_\alpha^{(i)}]e_\alpha^{(i)}\right)}_{\text{Claim: this is ZERO!}}$$

if $\alpha=\alpha_k \Rightarrow i=1 \Rightarrow [e_k,f_{\alpha_k}^{(1)}]=h_k\cdot d_k^{-1}=h_{\alpha_k}$ as we need $(e_\alpha,f_\alpha)=1$

To prove this claim, it suffices to verify that if $\alpha\in\Delta_+$ is s.t. $\alpha+\alpha_k\in\Delta_+$, then we have:

$$\boxed{\sum_i f_\alpha^{(i)}\otimes[e_\alpha^{(i)},e_k]=\sum_j [e_k,f_{\alpha+\alpha_k}^{(j)}]\otimes e_{\alpha+\alpha_k}^{(j)}}$$

Now we argue as in [Lecture 13, Lemmas 2, 2']:

$$[e_\alpha^{(i)},e_k]=\sum_j([e_\alpha^{(i)},e_k],f_{\alpha+\alpha_k}^{(j)})\cdot e_{\alpha+\alpha_k}^{(j)} \Rightarrow \sum_i f_\alpha^{(i)}\otimes[e_\alpha^{(i)},e_k]=\sum_j\Big(\sum_i f_\alpha^{(i)}\cdot\underbrace{([e_\alpha^{(i)},e_k],f_{\alpha+\alpha_k}^{(j)})}_{=(e_\alpha^{(i)},[e_k,f_{\alpha+\alpha_k}^{(j)}])}\Big)\otimes e_{\alpha+\alpha_k}^{(j)}$$
$$=\sum_j [e_k,f_{\alpha+\alpha_k}^{(j)}]\otimes e_{\alpha+\alpha_k}^{(j)}. \quad \checkmark$$

Thus: $[\Delta,e_k]=[\Delta_++\Delta_0,e_k]=0$.

Applying completely analogous argument, we also get $[\Delta,f_k]=0$.

Exercise: For $\mathfrak{g}(A)=\hat{\mathfrak{g}}$ ($\mathfrak{g}$-simple f.d.) show that $\Delta=2(k+h^\vee)(L_0+d)$, L_0 – 0^{th} Sugawara operator

* Locally finite and integrable modules

Now we shall develop the repr. theory. We start from the following definition:

Def 3: (a) For a Lie alg. $\mathfrak{g}$ and its module V, we say that $v\in V$ is *of finite type* if $\dim(U(\mathfrak{g})v)<\infty$

(b) We say that V is *locally finite* if every $v\in V$ is of finite type

Exercise: V is locally finite iff V is a sum of fin. dim. $\mathfrak{g}$-modules.

Def 4: A module V over a Kac-Moody algebra $\mathfrak{g}(A)$ is *integrable* if it is locally finite under each $sl_2^{(i)}$ (generated by e_i, h_i, f_i)

Rmk 2: Note that an sl_2-module M is loc. finite iff $M\simeq\oplus L_n$ ($n\in\mathbb{Z}_{\geq 0}$) and all such modules can be lifted (integrated) to an action of SL_2, hence, the terminology.

Prop 1: $\mathfrak{g}=\mathfrak{g}(A)$ is an integrable module over itself.

▶ ◦ First, we claim that each $f_j\in\mathfrak{g}(A)$ is of finite type.

Indeed, for $i=j$: the $sl_2^{(i)}$-module generated by f_j is 3-dimensional.

for $i\neq j$: the $sl_2^{(i)}$-module generated by f_j is $(1-a_{ij})$-dimensional, due to Serre rel-s.

◦ Completely analogously: all $e_j\in\mathfrak{g}(A)$ are also of finite type.

◦ As $\mathfrak{g}(A)$ is generated by $\{e_j,f_j\}$, it remains to notice an obvious fact:

if $x,y\in\mathfrak{g}$ are of finite type $\Rightarrow$ so is $[x,y]$. ■

Prop 2: A $\mathfrak{g}(A)$-module V is integrable iff there is a set $\{v_j\}_{j\in J}$ of generators of V over $\mathfrak{g}(A)=\mathfrak{g}$ such that each v_j is of finite type over each $sl_2^{(i)}$.

▶ $\Rightarrow$: Obvious as we may take for $\{v_j\}_{j\in J}$ all el-s of V.

$\Leftarrow$: Let $v\in V$ and $i\in\{1,\dots,r\}$. We want to show that v is of finite type over $sl_2^{(i)}$.

We may choose $v_1,\dots,v_N$ from our generating set, so that $v\in U(\mathfrak{g})v_1+\dots+U(\mathfrak{g})v_N=:V'\subseteq V$.

Let $W\subseteq V'$ be a f.dim. $sl_2^{(i)}$-submodule s.t. $v_1,\dots,v_N\in W$ (such exists as all v_i - of fin. type).

Then: there is a surjective $sl_2^{(i)}$-morphism $U(\mathfrak{g})\otimes W\twoheadrightarrow V'$, $x\otimes w\mapsto xw$

restriction of adjoint $\mathfrak{g}$-action to $sl_2^{(i)}\subseteq\mathfrak{g}$.

It suffices to show that $U(\mathfrak{g})$ is loc. finite $sl_2^{(i)}$-module (as then $U(\mathfrak{g})\otimes W$ - loc. fin $\Rightarrow V'$ - loc fin.)

But: by PBW $U(\mathfrak{g})\simeq S(\mathfrak{g})$ as $\mathfrak{g}$-modules and $S(\mathfrak{g})=\bigoplus_{k\geq 0}S^k(\mathfrak{g})$, $S^k(\mathfrak{g})\subseteq\mathfrak{g}^{\otimes k}$ - loc. finite $sl_2^{(i)}$-mod as a tensor product of loc.-finite (see Prop 1). ✓ ■

Prop 3: Let L_λ be the irreducible highest weight module over $\mathfrak{g}(A)$. Then:

$$\boxed{L_\lambda\text{ - integrable}\iff\lambda(h_i)\in\mathbb{Z}_{\geq 0}\ \ \forall\ 1\leq i\leq r.}$$

▶ $\Rightarrow$: If L_λ-integrable $\Rightarrow v_\lambda$ is of finite type for each $sl_2^{(i)}$, $1\leq i\leq r$.

But $e_i(v_\lambda)=0$, $h_i(v_\lambda)=\lambda(h_i)v_\lambda\xRightarrow{sl_2\text{-theory}}U(sl_2^{(i)})v_\lambda$ is fin.dim. only if $\lambda(h_i)\in\mathbb{Z}_{\geq 0}$.

$\Leftarrow$: If $\lambda(h_i)\in\mathbb{Z}_{\geq 0}$, then $v'_\lambda:=f_i^{1+\lambda(h_i)}v_\lambda$ satisfies $e_j(v'_\lambda)=0\ \forall j$, $v'_\lambda\in L_\lambda[\lambda-(1+\lambda(h_i))\alpha_i]$

Indeed: for $j\neq i$ this is immediate as $[e_j,f_i]=0$ and $e_j(v_\lambda)=0$

for $j=i$ this follows from sl_2-theory.

But then if $v'_\lambda\neq 0$, it would generate a proper graded submodule of $L_\lambda\Rightarrow$ ↯.

Hence: $v'_\lambda=0\Rightarrow v_\lambda$ is of fin. type over $sl_2^{(i)}$ $\forall i$.

Applying Prop 2, we see that L_λ - integrable ■

Write $\lambda\in P_+$

Def 5: The weights λ s.t. $\lambda(h_i)\in\mathbb{Z}_{\geq 0}$ $\forall i$ are called dominant integral weights (for $\mathfrak{g}(A)$ or $\mathfrak{g}_{ext}(A)$)

Rmk 3: If $\mathfrak{g}=\mathfrak{g}(A)$ - simple f.d, then L_λ-integrable $\iff L_\lambda$ - f. dim!

Goal: Find $ch(L_\lambda)$, $\lambda\in P_+$, generalizing the Weyl character formula for $\mathfrak{g}=\mathfrak{g}(A)$ – simple f.dim.

* Weyl group of $\mathfrak{g}(A)$ – Kac-Moody

Recall $P=\mathfrak{h}^*\oplus F$, endowed with a symm. nondeg. form $(\cdot,\cdot)$. Assume $\mathfrak{g}(A)$ – Kac-Moody alg.

‖ **Def 6:** For $i\in\{1,\dots,r\}$, define a linear map $r_i: P\to P$ via $\boxed{r_i(x)=x-x(h_i)\alpha_i}$

here we view x as an elt of $\mathfrak{h}^*_{ext}\simeq P$.

Exercise: $\forall x\in P\simeq \mathfrak{h}^*_{ext}$, $x(h_i)=d_i\cdot(x,\alpha_i)$

Lemma 1: (a) $r_i^2=Id$

(b) $(r_ix, r_iy)=(x,y)$ $\forall x,y\in P$.

▶ (a) $r_i^2(x)=r_i(x-x(h_i)\alpha_i)=x-x(h_i)\alpha_i-(x(h_i)-x(h_i)\underbrace{\alpha_i(h_i)}_{=2})\alpha_i = x$. (note: by above exercise, $\alpha_i(h_i)=d_i\cdot(\alpha_i,\alpha_i)=d_i\cdot\frac{a_{ii}}{d_i}=2$)

(b) $(r_ix,r_iy)=(x-x(h_i)\alpha_i,\ y-y(h_i)\alpha_i)=(x,y)-x(h_i)\cdot(y,\alpha_i)-y(h_i)\cdot(x,\alpha_i)+x(h_i)y(h_i)\cdot(\alpha_i,\alpha_i)$.

May be replaced by applying Exercise to each term

Claim: $x(h_i)y(h_i)\cdot(\alpha_i,\alpha_i)=x(h_i)\cdot(y,\alpha_i)+y(h_i)\cdot(x,\alpha_i)$.

- If $x,y\in\mathfrak{h}^*\subseteq P \Rightarrow \left.\begin{array}{l}(y,\alpha_i)=y(h_{\alpha_i})=d_i^{-1}\cdot y(h_i)\\ (x,\alpha_i)=x(h_{\alpha_i})=d_i^{-1}\cdot x(h_i)\end{array}\right\}\Rightarrow RHS=x(h_i)y(h_i)\cdot\frac{2}{d_i}=LHS$ as $(\alpha_i,\alpha_i)=\frac{a_{ii}}{d_i}\overset{\mathfrak{g}(A)\text{–Kac-Moody!}}{=}\frac{2}{d_i}$
- If $x\in\mathfrak{h}^*\subseteq P$, $y=\alpha_j\in P \Rightarrow \left.\begin{array}{l}(x,\alpha_i)=d_i^{-1}\cdot x(h_i)\\ (y,\alpha_i)=(\alpha_j,\alpha_i)=\frac{a_{ij}}{d_i}\\ y(h_i)=a_{ij}\end{array}\right\}\Rightarrow RHS=x(h_i)\cdot a_{ij}\cdot\frac{2}{d_i}=LHS$ as above
- If $x=\alpha_j$, $y=\alpha_k\in P\Rightarrow$ same argument

$\blacksquare$

Lemma 2: If V is an integrable $\mathfrak{g}(A)$-module, then $\forall\mu\in P$ $\exists$ isom. $V[\mu]\xrightarrow{\sim}V[r_i(\mu)]$.
In particular, $\dim(V[\mu])=\dim(V[r_i(\mu)])$

▶ As V is integrable $\xRightarrow{\mathfrak{sl}_2\text{-theory}}$ $\mu(h_i)\in\mathbb{Z}$. Moreover, $(r_i\mu)(h_i)=(\mu-\mu(h_i)\alpha_i)(h_i)=-\mu(h_i)$.
Thus, we may assume WLOG that $\mu(h_i)\in\mathbb{Z}_{\geq0}$. Then, the $\mathfrak{sl}_2$-theory implies that
$f_i^{\mu(h_i)}: V[\mu]\rightleftarrows V[r_i(\mu)]: e_i^{\mu(h_i)}$ are mutually inverse isomorphisms $\blacksquare$

‖ **Def 7:** The Weyl group of $\mathfrak{g}(A)$ is the subgroup W of $GL(P)$ generated by simple reflections r_i

Rmk 4: (a) r_i is a reflection in a codim=1 subspace given as $\mathrm{Ker}(\alpha_i,-)$.

(b) Note that $r_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$ $\forall i,j$

(c) In particular, $F\subseteq P$ is W-invariant, while $W\curvearrowright P/F$ is trivial.
For that reason, some authors view $W\subseteq GL(F)$.

Lemma 3: (a) The form $(\cdot,\cdot)$ on P is W-invariant.

(b) If V-integrable, $\mu\in P$, $w\in W$, then $V[\mu]\simeq V[w(\mu)]$.

(c) The set Δ of roots of $\mathfrak{g}=\mathfrak{g}(A)$ is W-invariant, and $\dim\mathfrak{g}_\alpha=\dim\mathfrak{g}_{w\alpha}$.

(d) $r_i(\alpha_i)=-\alpha_i$ and r_i permutes elements of $\Delta_+\setminus\{\alpha_i\}$.

▶ (a) Follows from Lemma 1(b).

(b) Follows from Lemma 2.

(c) Follows from (b), since adjoint repr-n is integrable by Prop 1.

(d) If $\alpha\in\Delta_+\setminus\{\alpha_i\}$, then $\alpha=\sum k_j\alpha_j$, $k_j\in\mathbb{Z}_{\geq0}$ and $\exists j'\neq i$ s.t. $k_{j'}>0$ (no roots $=m\cdot\alpha_i$ for $m>1$!)
But then $r_i(\alpha)$ has the same coeff. of $\alpha_{j'}$ $\Rightarrow r_i(\alpha)\in\Delta_+$. As $r_i^2=Id\Rightarrow r_i$ permutes $\Delta_+\setminus\{\alpha_i\}$ $\blacksquare$

* Weyl-Kac character formula

Theorem 2: Let $\lambda\in P_+$ - be a dominant integral weight of Kac-Moody algebra $\mathfrak{g}(A)$.
Let V be an integrable highest weight module (over $\mathfrak{g}_{ext}(A)$ or $\mathfrak{g}(A)$) with h.wt. λ.
(note: V is automatically in cat. $\mathcal{O}$!)

Then: (1) $V\simeq L_\lambda$

(2) The character of V is given by the following Weyl-Kac character formula:

$$\boxed{ch(V)=\sum_{w\in W}\det(w)\, ch\, M_{w(\lambda+\rho)-\rho} = \sum_{w\in W}\frac{\det(w)\cdot e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})^{\dim\mathfrak{g}_\alpha}}}$$

this follows immediately from the f-la for $ch\, M_\lambda$.

where:

- $\rho\in\mathfrak{h}^*$ is defined as before
- $\det(w)\in\{\pm1\}$ is given by $\det(r_{i_1}\cdots r_{i_k})=(-1)^k$ which is clearly well-defined.

Rmk 5: (a) We always have an epimorphism $V\twoheadrightarrow L_\lambda$ for V - h.wt. module of h.wt. λ.
As L_λ ($\lambda\in P_+$) is integrable, we see that (2) $\Rightarrow$ (1)!

(b) Define $L'_\lambda := M_\lambda/\langle f_i^{1+\lambda(h_i)}v_\lambda\rangle_{i=1}^{r}$. Then the same argument as was used in the proof of Prop 3 shows that L'_λ is integrable. Hence, by (a):
For $\lambda\in P_+$, have: $\boxed{L'_\lambda\simeq L_\lambda}$, i.e. L_λ is defined by the relations $f_i^{1+\lambda(h_i)}v_\lambda=0\ \forall i$.

(c) As $L_{\lambda=0}\simeq\mathbb{C}$ - trivial module $\Rightarrow$ $ch\, L_0=1$. Hence, we get:

$$\boxed{\sum_{w\in W}\det(w)e^{w\rho-\rho}=\prod_{\alpha>0}(1-e^{-\alpha})^{\dim\mathfrak{g}_\alpha}}$$ $\leftarrow$ Weyl-Kac denominator formula

(d) When $\mathfrak{g}\simeq\mathfrak{g}(A)$ - simple f.dim, then $\dim\mathfrak{g}_\alpha=1\ \forall\alpha\in\Delta\Rightarrow$
$\Rightarrow$ recover the Weyl character formula.

(e) Due to (c), for $\lambda\in P_+$ we have

$$\boxed{ch\, L_\lambda=\frac{\sum_{w\in W}\det(w)\,e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W}\det(w)\,e^{w\rho-\rho}}}$$

Set $\boxed{K:=e^\rho\prod_{\alpha>0}(1-e^{-\alpha})^{\dim\mathfrak{g}_\alpha}}$. Note that the Weyl group naturally acts on $\mathcal{R}$. $\leftarrow$ the ring where all $ch(M)$ belong to.

Lemma 4: K is W-anti-invariant, i.e. $wK=\det(w)\cdot K$

▸ $K=e^\rho(1-e^{-\alpha_i})\prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}(1-e^{-\alpha})^{\dim\mathfrak{g}_\alpha}$. According to Lemma 3(c,d): $r_i\Big(\prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}(1-e^{-\alpha})^{\dim\mathfrak{g}_\alpha}\Big)=\prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}(1-e^{-\alpha})^{\dim\mathfrak{g}_\alpha}$.

Also: $r_i(e^\rho(1-e^{-\alpha_i}))=e^{\rho-\rho(h_i)\alpha_i}\cdot(1-e^{\alpha_i})=e^\rho(e^{-\alpha_i}-1)=-e^\rho(1-e^{-\alpha_i})$.

Hence: $r_iK=-K\ \forall i\Rightarrow\boxed{wK=\det(w)\cdot K}\ \forall w\in W$ ∎

We will prove Theorem 2 next time!