— LECTURE 19 —

* Last time: We formulated the following Theorem:

Weyl-Kac Character Formula: Let $\chi \in P_+$ be a dominant integral weight of Kac-Moody alg. $\mathfrak{g}(A)$.
Let $V$ be an integrable h.wt. repr-n with h. weight $\chi$. Then:

$$\boxed{ch\, V = \sum_{w\in W} \det(w)\cdot ch\, M_{w(\chi+\rho)-\rho} = \sum_{w\in W} \frac{\det(w)\cdot e^{w(\chi+\rho)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}}$$

* Let us now prove this theorem.

Lemma 1: Let $\chi \in P_+$. Then:

a) $W\chi \subseteq D(\chi)$, where $D(\chi) = \{\chi - \sum n_i\alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$

b) If $D \subseteq D(\chi)$ is a $W$-invariant set, then $D \cap P_+ \neq \emptyset$.

a) Consider the $\mathfrak{g}(A)$-module $L_\chi$. Last time: $\chi \in P_+ \Rightarrow L_\chi$ - integrable $\Rightarrow P(L_\chi) := \{\text{weights of } L_\chi\}$ - $W$-invariant set.
As $\chi \in P(L_\chi) \overset{W\text{-inv}}{\Longrightarrow} W\chi \subseteq P(L_\chi)$. But: $P(L_\chi) \subseteq D(\chi) \Rightarrow \boxed{W\chi \subseteq D(\chi)}$

b) Let $\psi \in D$. Choose $w \in W$ so that in the expression $\chi - w\psi = \sum k_i\alpha_i$ ($k_i \in \mathbb{Z}_{\geq 0}$), the sum $\sum k_i$ is minimal. We claim that $w\psi \in P_+$. Assume not, i.e. $\exists i$: $\underset{\text{equiv. } w\psi(h_i)<0}{(w\psi, \alpha_i) < 0} \Rightarrow r_i(w\psi) = w\psi - \underbrace{w\psi(h_i)}_{m_i<0}\cdot\alpha_i$

Then: $\chi - r_i(w\psi) = \sum k_j\alpha_j + m_i\alpha_i \Rightarrow \sum k_j + m_i < \sum k_j \Rightarrow$ Contradiction with minimality! $\Rightarrow w\psi \in P_+$ $\blacksquare$

Cor 1: If $w \in W\setminus\{1\}$, then $\exists i$ s.t. $w\alpha_i < 0$.

Choose $\chi \in P_+$ s.t. $w\chi \neq \chi$ (existence of such $\chi$ follows from $w \neq 1$ ← Exercise!)
Then $w^{-1}\chi = \chi - \sum k_i\alpha_i$ for some $k_i \in \mathbb{Z}_{\geq 0}$ by Lemma 1(a).

Hence: $\chi = w(w^{-1}\chi) = w\chi - \sum k_i w(\alpha_i) \overset{\text{Lemma 1a}}{=\!=\!=} (\chi - \sum k_i'\alpha_i) - \sum k_i w(\alpha_i)$ with $k_i', k_i \in \mathbb{Z}_{\geq 0}$.

$\left.\begin{array}{l} \Rightarrow \sum k_i'\alpha_i + \sum k_i w\alpha_i = 0 \\ \text{But: } w\chi \neq \chi \Rightarrow \text{At least one } k_i' > 0 \end{array}\right\} \Rightarrow$ it may not happen that all $w\alpha_i > 0 \Rightarrow \exists i: w(\alpha_i) < 0$. $\blacksquare$

Lemma 2: Let $\varphi, \psi \in P$ satisfy $\varphi(h_i) > 0$, $\psi(h_i) \geq 0$ $\forall i$. Then $w\varphi = \psi \Leftrightarrow w = \mathrm{Id}$, $\varphi = \psi$.

$\varphi(h_i) > 0 \Longleftrightarrow (\varphi, \alpha_i) > 0$ $\forall i$.

If $w \neq 1$, then by Cor 1: $\exists i$ s.t. $w\alpha_i < 0$.

Hence: $0 < (\varphi, \alpha_i) = (w^{-1}\psi, \alpha_i) \overset{W\text{-inv.}}{=\!=\!=} (\psi, w\alpha_i) \leq 0 \Rightarrow$ Contradiction! $\Rightarrow w = 1 \Rightarrow \varphi = \psi$ $\blacksquare$

Last time we also proved:

Lemma 3: $wK = \det(w)\cdot K$ $\forall w \in W$, where $K = e^\rho \prod_{\alpha>0}(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$.

We shall also need the following simple result:

Lemma 4: Let $\mu, \nu \in P_+$ be such that $\mu \in D(\nu)$, $\mu \neq \nu$. Then $(\nu+\rho, \nu+\rho) - (\mu+\rho, \mu+\rho) > 0$.

Let $\nu - \mu = \sum k_i\alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$ and $\exists i: k_i > 0$. Then:

$(\nu+\rho, \nu+\rho) - (\mu+\rho, \mu+\rho) = (\nu,\nu) - (\mu,\mu) + (2\nu, \rho) - (2\mu, \rho) = (\nu-\mu, \nu+\mu+2\rho) = \sum_i k_i \underbrace{(\nu+\mu+2\rho, \alpha_i)}_{>0} > 0$ $\blacksquare$

Recall that ch $M_\lambda$ form a topological basis of $R$. More precisely, we have:

<u>Lemma 5</u>: For any $V\in\mathcal{O}$, we have $\boxed{ch(V)=\sum_{\chi} c_\chi\cdot ch\, M_\chi \text{ with } c_\chi\in\mathbb{Z} \text{ and } \chi\in\bigcup_{\lambda\in P(V)} D(\lambda)}$

▶ As $V\in\mathcal{O}\Rightarrow \exists\, \lambda_1,\dots,\lambda_m\in P$ s.t. $P(V)\subseteq\bigcup_{i=1}^{m} D(\lambda_i)$.

For $\lambda\in D(\lambda_i)$ written as $\lambda=\lambda_i-\sum k_i\alpha_i$ ($k_i\in\mathbb{Z}_{\geq 0}$), we set $h_i(\lambda):=\sum k_i$.

For $\lambda\in\bigcup_{i=1}^{m} D(\lambda_i)$, set $h(\lambda):=\sum_i h_i(\lambda)$, where the sum is over those $i$ s.t. $\lambda\in D(\lambda_i)$

Finally, we set $h(V):=\min_{\lambda\in P(V)} h(\lambda)$. Let $\mu_1,\dots,\mu_r$ be all elts of $P(V)$ with $h(\mu_i)=h(V)$, and $\{v_{i,1},\dots,v_{i,k_i}\}$ be a basis of $V[\mu_i]$.

As each $v_{i,j}$ is clearly a highest weight vector, we have a morphism of modules

$\boxed{\varphi:\bigoplus_i M_{\mu_i}^{\oplus k_i}\longrightarrow V \quad \text{sending } k_i \text{ copies of } v_{\mu_i}\in M_{\mu_i} \text{ to } v_{i,1},\dots,v_{i,k_i}}$

Let $K:=\mathrm{Ker}(\varphi)$, $C:=\mathrm{Coker}(\varphi)$, so that we get an exact sequence:

$$\boxed{0\to K\to\bigoplus_i M_{\mu_i}^{\oplus k_i}\xrightarrow{\varphi} V\to C\to 0}$$

Then: a) $ch(V)=\sum k_i\cdot ch\, M_{\mu_i}-ch(K)+ch(C)$

b) $P(K),P(C)\subseteq\bigcup_{i=1}^{m} D(\lambda_i)$

c) $h(K),h(C)>h(V)$

Thus applying this argument to $K,C$ again and proceeding further (<u>may not terminate</u>!), we obtain the result. ■

Let us now get to the proof of the Weyl-Kac Theorem.

- Applying Lemma 5 (and its proof) to $V$ of theorem, we get $\boxed{ch\, V=\sum_{\psi\in D(\chi)} c_\psi\cdot ch\, M_\psi,\ c_\chi=1.}$
- Next, we obtain a restriction on $\psi$ s.t. $c_\psi\neq 0$:

<u>Lemma 6</u>: If $c_\psi\neq 0$, then $(\psi+\rho,\psi+\rho)=(\chi+\rho,\chi+\rho)$

▶ The action of the Casimir operator $\Delta$ on $M_\psi$ and $V$ is by $\begin{matrix}(\psi,\psi+2\rho)=\\(\psi+\rho,\psi+\rho)-(\rho,\rho)\end{matrix}$ and $\begin{matrix}(\chi,\chi+2\rho)=\\(\chi+\rho,\chi+\rho)-(\rho,\rho).\end{matrix}$

Following our proof of Lemma 5 and recalling that $\Delta$ commutes with $\mathfrak{g}(A)$-action, we immediately get $(\psi+\rho,\psi+\rho)=(\chi+\rho,\chi+\rho)$ (otherwise no nontrivial hom. $M_\psi\to V$ exists) ■

<u>Note</u>: Both $K,C$ have the same action of $\Delta$ as $M_\psi$ and $V$.

- Next, we compute $c_\psi$ for $\psi=w(\chi+\rho)-\rho$.

<u>Lemma 7</u>: If $\psi+\rho=w(\chi+\rho)$, $w\in W$, then $c_\psi=\det(w)\cdot c_\chi$

▶ Applying Lemma 5 with f-la for ch $M_\chi$, we get $ch(V)\cdot K=\sum_\psi c_\psi\cdot e^{\psi+\rho}$

But $ch(V)$ − $W$-invariant (as a character of an integrable module), $K$ − $W$-antiinvariant (Lemma 3) $\Rightarrow$

$\Rightarrow\sum_\psi c_\psi e^{\psi+\rho}$ is $W$-anti-invariant $\Rightarrow \det(w)\cdot\sum_\psi c_\psi e^{\psi+\rho}=\sum c_{w'\circ\psi}\cdot e^{\psi+\rho}$, where $w'\circ\psi=w'(\psi+\rho)-\rho$

$\Rightarrow c_{w'\circ\psi}=\det(w)\cdot c_\psi \Rightarrow c_{w\circ\psi}=\det(w)\cdot c_\psi$ ■

- Finally, let us describe $D:=\{\psi\in P\mid c_{\psi-\rho}\neq 0\}$.

<u>Lemma 8</u>: $D = W(\chi+\rho)$ — $W$-orbit of $\chi+\rho$.

By Lemma 7: $W(\chi+\rho) \subseteq D$. By the proof of Lemma 7, it is clear that $D$ is $W$-invariant.
But due to Lemma 1(b): $D \cap P_+ \neq \emptyset$ and $(D - W(\chi+\rho)) \cap P_+ \neq \emptyset$ if $D \neq W(\chi+\rho)$.
In the latter case: $\exists\, \beta \in (D - W(\chi+\rho)) \cap P_+$. By definition of $D$, we get $\beta-\rho \in D(\chi)$.
Now let us apply Lemma 4 (or its slight upgrade) for $\nu=\chi$, $\mu=\beta-\rho$:

$$(\chi+\rho, \chi+\rho) - (\beta, \beta) = \sum_i k_i \underbrace{(\chi+\beta+\rho, \alpha_i)}_{>0} > 0$$

However, this contradicts Lemma 6 which asserts $(\chi+\rho, \chi+\rho) = (\beta, \beta)\ \forall \beta \in D$.
Contradiction! So $W(\chi+\rho) = D$.

- According to Lemma 2, the map $W \to P$, $w \mapsto w(\chi+\rho)$, is injective.
Combining this with Lemmas 7–8 and $c_\chi = 1$, we get:

$$\boxed{\mathrm{ch}(V) = \sum_{w \in W} \det(w) \cdot \mathrm{ch}\, M_{w(\chi+\rho)-\rho}}$$

This completes our proof of the Weyl–Kac character f-la!

* * *

* <u>Weyl–Kac formula for $\mathfrak{g}(A) = \tilde{\mathfrak{g}}$ (particular case of interest: $\mathfrak{g} = \mathfrak{sl}_2$)</u>

- Let $\mathfrak{g}$ be a simple f.d. Lie algebra $\rightsquigarrow L\mathfrak{g} = \mathfrak{g}[t,t^{-1}]$, $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$, $\tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$.

[<u>Note</u>: The extension $\tilde{\mathfrak{g}}$ is smaller than $\mathfrak{g}^{\mathrm{ext}}(\hat{A})$, but is equivalent for the purpose of defining category $\mathcal{O}$, Verma modules $M_\lambda$, irreducibles $L_\lambda$, and even Weyl–Kac char. f-la!]

Similarly to the case of $\mathfrak{gl}_n$ (discussed in Lecture 12), the algebra $\tilde{\mathfrak{g}}$ can be endowed with an invariant symmetric nondegenerate form as follows:

- For $a(t), b(t) \in L\mathfrak{g}$, set $(a(t), b(t)) := \mathrm{Res}_{t=0}\, (a(t), b(t)) \frac{dt}{t}$
- Also set $(K,d) = 1$, $(K,K) = (d,d) = (K, L\mathfrak{g}) = (d, L\mathfrak{g}) = 0$.

this is the standard pairing on $\mathfrak{g}$ applied coefficient-wise; $(\theta,\theta)=2$

As in [Lecture 12, Lemma 2], we have:

<u>Lemma 9</u>: Thus defined symmetric pairing on $\tilde{\mathfrak{g}}$ is invariant & nondegenerate.

<u>Exercise</u>: Prove that any symm. inv. nondeg. form on $\tilde{\mathfrak{g}}$ with $(d,d)=0$ is a multiple of the above one.

- Let $G$ be the simply connected $\mathbb{C}$-Lie group with $\mathrm{Lie}\, G = \mathfrak{g}$ (e.g. $\mathfrak{g} = \mathfrak{sl}_n \rightsquigarrow G = SL_n$).
$G$ may be realized as an alg. subgp $\subseteq \mathrm{Mat}_{N\times N}(\mathbb{C})$ by certain algebraic equations
$\rightsquigarrow \boxed{LG = G[t,t^{-1}]}$ — defined by the same alg. equations in $\mathrm{Mat}_{N\times N}(\mathbb{C}[t,t^{-1}])$.

Note $G \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g} \rightsquigarrow LG \overset{\mathrm{Ad}}{\curvearrowright} L\mathfrak{g}$.

<u>Lemma 10</u>: The action $LG \curvearrowright L\mathfrak{g}$ uniquely extends to $LG \curvearrowright L\mathfrak{g} \oplus \mathbb{C}d$ via

$$\boxed{g(t)(d) = d + g(t) \cdot t\frac{d}{dt}(g^{-1}(t)) = d - t \cdot g'(t) \cdot g(t)^{-1}}$$

<u>Note</u>: Viewing $g(t)$ as an algebraic map $\mathbb{C}^\times \to G$, it is clear that $\forall t_0 \in \mathbb{C}^\times$: $g'(t_0) g(t_0)^{-1} \in \mathfrak{g}$
<u>Exercise</u>: Explain!

- For $g \in LG$, $a \in L\mathfrak{g}$: $[g(d), g(a)] = [d - tg' \cdot g^{-1}, gag^{-1}] \overset{[d, b(t)] = t\frac{d}{dt}b(t)}{=\!=\!=} t\partial_t(gag^{-1}) - t \cdot g'ag^{-1} + t\, gag^{-1}g'g^{-1}$
$= \cancel{tg'ag^{-1}} + tga'g^{-1} + \cancel{tga\partial_t(g^{-1})} - \cancel{tg'ag^{-1}} - \cancel{tga\partial_t(g^{-1})} = g(ta') = g([d,a])$

- Also clear: $(g(t)h(t))(d) = g(t)(h(t)d)$.

Thus, the action $LG \curvearrowright L\mathfrak{g}$ extends to $LG \curvearrowright L\mathfrak{g}\oplus\mathbb{C}d$

meaning that two $LG$-actions are compatible under the Lie alg. homom. $L\mathfrak{g}\hookrightarrow L\mathfrak{g}\oplus\mathbb{C}d$.

The latter can be upgraded to $LG \curvearrowright \tilde{\mathfrak{g}}$

meaning that two $LG$-actions are compatible under the Lie alg. homom. $\tilde{\mathfrak{g}} \twoheadrightarrow L\mathfrak{g}\oplus\mathbb{C}d$ $(K\mapsto 0)$

**Prop 1**: There is an action of $LG$ on $\tilde{\mathfrak{g}}$ given by

$$\boxed{\begin{aligned} &g(K)=K\\ &g(a)=gag^{-1}+\mathrm{Res}_{t=0}\, tr(g'ag^{-1})dt\cdot K\\ &g(d)=d-tg'g^{-1}-\tfrac{1}{2}\mathrm{Res}_{t=0}(tg'g^{-1},tg'g^{-1})\tfrac{dt}{t}\cdot K\end{aligned}}$$

Here $g\in LG$, $a\in L\mathfrak{g}$ and viewing $G\subseteq Mat_{N\times N}(\mathbb{C}) \Rightarrow \mathfrak{g}\subseteq\mathfrak{gl}_N \Rightarrow tr(X)$ ($X\in\mathfrak{g}$) is well-defined and may assume $(X,Y)=tr(XY)$ ($X,Y\in\mathfrak{g}$).

Moreover, this action preserves the invariant nondeg. form on $\tilde{\mathfrak{g}}$.

- First, let us check that each $g\in LG$ defines a Lie alg. autom. of $\tilde{\mathfrak{g}}$. There are only 2 key verifications:
  - $a,b\in L\mathfrak{g} \Rightarrow [a,b]_{\tilde{\mathfrak{g}}}=[a,b]_{L\mathfrak{g}}+\mathrm{Res}_{t=0}(a',b)dt\cdot K \Rightarrow g([a,b])=g[a,b]g^{-1}+\mathrm{Res}_{t=0}\, tr(g'[a,b]g^{-1}+a'b)dt\cdot K$

    $[g(a),g(b)]_{\tilde{\mathfrak{g}}}=[gag^{-1},gbg^{-1}]_{\tilde{\mathfrak{g}}}=\underbrace{[gag^{-1},gbg^{-1}]_{L\mathfrak{g}}}_{=g[a,b]_{L\mathfrak{g}}g^{-1}}+\mathrm{Res}_{t=0}\, tr\left(\tfrac{d}{dt}(gag^{-1})\cdot gbg^{-1}\right)dt\cdot K$

    But: $\mathrm{Res}_{t=0}\, tr\left(\tfrac{d}{dt}(gag^{-1})\cdot gbg^{-1}\right)dt=\mathrm{Res}_{t=0}\, tr\left((g'ag^{-1}+ga'g^{-1}-gag^{-1}g'g^{-1})\cdot gbg^{-1}\right)dt$

    $=\mathrm{Res}_{t=0}\, tr\left(g'abg^{-1}+ga'bg^{-1}-gag^{-1}g'bg^{-1}\right)dt=\mathrm{Res}_{t=0}\, tr\left(a'b+g'[a,b]g^{-1}\right)dt$

    $\Rightarrow \boxed{g([a,b])=[g(a),g(b)]}$

  - $a\in L\mathfrak{g} \Rightarrow [d,a]=ta' \Rightarrow g([d,a])=tga'g^{-1}+\mathrm{Res}_{t=0}\, tr(g'\cdot ta'\cdot g^{-1})dt\cdot K$

    $[g(d),g(a)]=[d-tg'g^{-1},gag^{-1}]_{\tilde{\mathfrak{g}}}=\underbrace{[d-tg'g^{-1},gag^{-1}]_{L\mathfrak{g}\oplus\mathbb{C}d}}_{=g(ta')g^{-1}\text{ by Lemma 10}}-\mathrm{Res}_{t=0}\, tr\left((tg'g^{-1})'\cdot gag^{-1}\right)dt\cdot K$

    But: $-\mathrm{Res}_{t=0}\, tr\left((tg'g^{-1})'\cdot gag^{-1}\right)dt=-\mathrm{Res}_{t=0}\, tr\left(g'ag^{-1}+tg''ag^{-1}-tg'g^{-1}g'ag^{-1}\right)dt$

    $=\mathrm{Res}_{t=0}\, tr(tg'a'g^{-1})dt-\mathrm{Res}_{t=0}\, tr\left((tg'ag^{-1})'\right)dt=\mathrm{Res}_{t=0}\, tr(tg'a'g^{-1})dt$

    $\Rightarrow \boxed{g([d,a])=[g(d),g(a)]}$

- **Exercise**: Verify that above map $LG\to Aut_{\text{Lie-alg}}(\tilde{\mathfrak{g}})$ is indeed a group homomorphism.
- Finally, we need to check that this action preserves the inv. nondeg. pairing on $\tilde{\mathfrak{g}}$. There are only two nontrivial verifications:
  - $(g(d),g(a))=(d-tg'g^{-1}-\ldots K,\ gag^{-1}+\mathrm{Res}_{t=0}\, tr(g'ag^{-1})dt\cdot K)=\mathrm{Res}_{t=0}\, tr(g'ag^{-1})dt-\mathrm{Res}_{t=0}\, tr(tg'g^{-1}\cdot gag^{-1})\frac{dt}{t}$

    $=0=(d,a)$
  - $(g(d),g(d))=\left(d-tg'g^{-1}-\tfrac{1}{2}\mathrm{Res}_{t=0}(tg'g^{-1},tg'g^{-1})\tfrac{dt}{t}\cdot K,\ d-tg'g^{-1}-\tfrac{1}{2}\mathrm{Res}_{t=0}(tg'g^{-1},tg'g^{-1})\tfrac{dt}{t}K\right)$

    $=(tg'g^{-1},tg'g^{-1})-\mathrm{Res}_{t=0}(tg'g^{-1},tg'g^{-1})\frac{dt}{t}=0=(d,d)$

∎

**Exercise**: (a) Prove that this is a unique lifting of $LG\curvearrowright L\mathfrak{g}\oplus\mathbb{C}d$ to $LG\curvearrowright\tilde{\mathfrak{g}}$ which preserves pairing

(b) Explain how to rewrite $tr(g'ag^{-1})$ more invariantly just using $(\cdot,\cdot)$.

From now on, we shall treat the simplest case of $\hat{\mathfrak g}$ with $\mathfrak g=\mathfrak{sl}_2$.

Identifying $\hat{\mathfrak h}$ with $\hat{\mathfrak h}^*$ using the nondeg. bilinear form (restriction of the inv. form on $\hat{\mathfrak g}$), we have $\hat{\mathfrak h}=\mathbb C\alpha\oplus\mathbb CK\oplus\mathbb Cd$. The pairing is: $\boxed{(\alpha,\alpha)=2,\ (K,d)=1,\ (K,K)=(K,\alpha)=(d,d)=(d,\alpha)=0.}$

The generators $h_0,h_1$ are explicitly given via $\boxed{h_1=\alpha,\ h_0=K-\alpha.}$

Recall the fundamental weights $\omega_0,\omega_1$ determined by $\omega_i(h_j)=\delta_{ij}$, $\omega_i(d)=0$. Explicitly: $\boxed{\omega_0=d,\ \omega_1=\frac{1}{2}\alpha+d}$. Any weight $\lambda$ can be written as $\boxed{\lambda=md+\frac{n}{2}\alpha+\tau K=(m-n)\omega_0+n\omega_1+\tau K.}$

Then:
1) $L_\lambda$ is integrable iff $m,n\in\mathbb Z_{\geq 0}$ and $m\geq n$
2) $L_\lambda$ is unitary iff $m,n\in\mathbb Z_{\geq 0}$, $m\geq n$, and $\tau\in\mathbb R$.
3) The level of $L_\lambda$ equals $m$.

We would like to write down very explicitly the Weyl-Kac character f-la for integrable $\hat{\mathfrak{sl}}_2$-module $L_\lambda$. To do that, we need to describe the Weyl group.

Motivated by the definition of the Weyl gp $W$ of alg. gps, $W=N(T)/Z(T)$, we introduce:

**Def 1**: Let $\widetilde W:=\{g\in LG\mid g\hat{\mathfrak h}g^{-1}\subseteq\hat{\mathfrak h}\}$ and $W$ be the image of $\widetilde W$ in $\mathrm{End}_{\mathbb C}(\hat{\mathfrak h})$.

**Fact**: The resulting $W$ coincides with the Weyl gp defined last time! (that's not quite trivial)

In the particular case of $\mathfrak g=\mathfrak{sl}_2$, we have (by Prop 1) $g(\alpha)=g\alpha g^{-1}+\mathrm{Res}_{t=0}\,\mathrm{tr}(g'\alpha g^{-1})dt\cdot K$, hence $g\in\widetilde W\Rightarrow g\alpha g^{-1}\in\mathfrak h=\mathbb C\alpha\Rightarrow g\alpha g^{-1}=\begin{pmatrix}\nu&0\\0&-\nu\end{pmatrix}$. But $\det(g\alpha g^{-1})=\det(\alpha)=-1\Rightarrow\nu=\pm1$.

- If $\nu=1\Rightarrow g\begin{pmatrix}1&0\\0&-1\end{pmatrix}g^{-1}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Rightarrow g=\begin{pmatrix}*&0\\0&*\end{pmatrix}\Rightarrow\boxed{g=\begin{pmatrix}a(t)&0\\0&a(t)^{-1}\end{pmatrix}}$
- If $\nu=-1\Rightarrow g\begin{pmatrix}1&0\\0&-1\end{pmatrix}g^{-1}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\Rightarrow\boxed{g=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot\begin{pmatrix}a(t)&0\\0&a(t)^{-1}\end{pmatrix}}$

Here $a(t),a(t)^{-1}\in\mathbb C[t,t^{-1}]$
$\Updownarrow$
$\boxed{a(t)=c\cdot t^k,\ c\in\mathbb C^*,\ k\in\mathbb Z}$

And it is clear that these el-s belong to $\widetilde W$ as $g(K)=K$, while

- If $g=\begin{pmatrix}ct^k&0\\0&c^{-1}t^{-k}\end{pmatrix}\Rightarrow tg'g^{-1}=t\cdot\begin{pmatrix}ckt^{k-1}&0\\0&-c^{-1}k\cdot t^{-k-1}\end{pmatrix}\begin{pmatrix}c^{-1}t^{-k}&0\\0&ct^k\end{pmatrix}=k\cdot\alpha\Rightarrow g(d)\in\hat{\mathfrak h}$
- If $g=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}ct^k&0\\0&c^{-1}t^{-k}\end{pmatrix}=\begin{pmatrix}0&-c^{-1}t^{-k}\\ct^k&0\end{pmatrix}\Rightarrow tg'g^{-1}=t\cdot\begin{pmatrix}0&+kc^{-1}t^{-k-1}\\kct^{k-1}&0\end{pmatrix}\cdot\begin{pmatrix}0&c^{-1}t^{-k}\\-ct^k&0\end{pmatrix}=-k\cdot\alpha\Rightarrow g(d)\in\hat{\mathfrak h}$

From these verifications, we also see that the images of these el-s in $W$ do not depend on $c\in\mathbb C^*$, but do depend on $k\in\mathbb Z$. As a result, we get the following description of the Weyl gp of $\hat{\mathfrak{sl}}_2$:

$$\boxed{W=\{t_k,\ r_\alpha t_k\mid k\in\mathbb Z\}}\quad\text{where } r_\alpha=\mathrm{Ad}\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad t_k=\mathrm{Ad}\begin{pmatrix}t^k&0\\0&t^{-k}\end{pmatrix}$$

the group structure is also obvious!

Explicitly: $\boxed{\begin{array}{l} r_\alpha:\alpha\mapsto-\alpha,\ K\mapsto K,\ d\mapsto d\\ t_k:\alpha\mapsto\alpha+2k\cdot K,\ K\mapsto K,\ d\mapsto d-k\cdot\alpha-k^2\cdot K\end{array}}\Rightarrow\begin{array}{l}\det(r_\alpha)=-1\\ \boxed{\det(t_k)=1}\end{array}\Rightarrow\boxed{\det(r_\alpha t_k)=-1}$

Next time we shall start by understanding what the Weyl-Kac character formula gives in the simplest case of $\hat{\mathfrak{sl}}_2$: the result will manifestly feature Jacobi-Riemann theta f-s!