— LECTURE 20 —

Weyl-Kac character fla: $$\mathrm{ch}_{L_\lambda} = \frac{\sum_{w\in W} \det(w) e^{w(\lambda+\rho)}}{\sum_{w\in W}\det(w)e^{w(\rho)}}$$ ← for any Kac-Moody $\mathfrak{g}(A)$.

For $h\in\hat{\mathfrak{h}}$, we can compute $\mathrm{ch}_{L_\lambda}(h)$ which is the same as formal $\mathrm{tr}_{L_\lambda}(e^h)$. Hence, we have

$$\mathrm{ch}_{L_\lambda}(h) = \frac{\sum_{w\in W}\det(w)e^{(w(\lambda+\rho),h)}}{\sum_{w\in W}\det(w)e^{(w(\rho),h)}} \qquad (*)$$

TODAY: $\mathfrak{g}(A)=\hat{\mathfrak{sl}}_2 \rightsquigarrow \hat{\mathfrak{sl}}_2$ instead of $\mathfrak{g}_{ext}(A)$.

Last time: $W=\{t_k, r_\alpha t_k\}_{k\in\mathbb{Z}}$, $\rho=\omega_0+\omega_1=\frac{1}{2}\alpha+2d$, $\det(t_k)=1$, $\det(r_\alpha t_k)=-1$.

We shall take a general el-t h of $\hat{\mathfrak{h}}$ written in the following form:

$$h=2\pi i\left(\tfrac{1}{2}z\alpha-\tau\cdot d+u\cdot K\right), \quad z,\tau,u\in\mathbb{C}$$ (as z,τ,u vary - get all el-ts of $\hat{\mathfrak{h}}$).

Then, plugging this h into RHS of $(*)$ we get a formal series in exponentials of z,τ,u, which turns out to converge to analytic functions in appropriate regions (see below!).

Def 1: Define theta functions

$$\Theta_{n,m}(\tau,z,u) := e^{2\pi i m u}\sum_{k\in\frac{n}{2m}+\mathbb{Z}} e^{2\pi i m(k^2\tau+kz)}$$ ← This sum converges absolutely for any z,u and $\mathrm{Im}(\tau)>0$.

Since numerator & denominator of RHS in $(*)$ are similar, we should compute $\sum_{w\in W}\det(w)e^{(w(\mu),h)}$ for $\mu=md+\frac{n}{2}\alpha+rK$.

- $w=t_k \Rightarrow \det(w)=1$, $w(\mu)=m(d-k\cdot\alpha-k^2\cdot K)+\frac{n}{2}(\alpha+2k\cdot K)+r\cdot K = m\cdot d+(\frac{n}{2}-mk)\cdot\alpha+(r+kn-k^2m)\cdot K$

$$(w(\mu),h)=2\pi i\left(z(\tfrac{n}{2}-mk)+mu-\tau(r+kn-k^2m)\right)$$

- $w=r_\alpha t_k \Rightarrow \det(w)=-1$, $w(\mu)=md-(\frac{n}{2}-mk)\cdot\alpha+(r+kn-k^2m)\cdot K$

$$(w(\mu),h)=2\pi i\left(-z(\tfrac{n}{2}-mk)+mu-\tau(r+kn-k^2m)\right)$$

So: $$\sum_{w\in W}\det(w)e^{(w(\mu),h)}=\sum_{k\in\mathbb{Z}}e^{2\pi i m u}\cdot\left(e^{2\pi i(z(\frac{n}{2}-mk)-\tau(r+kn-k^2m))}-e^{2\pi i(-z(\frac{n}{2}-mk)-\tau(r+kn-k^2m))}\right).$$

To relate this to the above Jacobi-Riemann theta functions, we:

- replace k by $\frac{n}{2m}-k$ ($k\in\frac{n}{2m}+\mathbb{Z}$ now) in the first sum (corresponding to t_k), to get $(w(\mu),h)=2\pi i(mu+mkz+(mk^2-(r+\frac{n^2}{4m}))\tau)$ ⇒ summing over all $k\in\frac{n}{2m}+\mathbb{Z}$, we get

$$\sum_{k\in\mathbb{Z}}\det(t_k)e^{(t_k(\mu),h)}=e^{2\pi i\tau\cdot(-(r+\frac{n^2}{4m}))}\cdot\Theta_{n,m}(\tau,z,u)=q^{-(r+\frac{n^2}{4m})}\cdot\Theta_{n,m}(\tau,z,u), \quad q:=e^{2\pi i\tau}$$

- replace k by $\frac{n}{2m}+k$ (with $k\in-\frac{n}{2m}+\mathbb{Z}$) in the second sum (corresp. to $r_\alpha t_k$), to get $(w(\mu),h)=2\pi i(mu+mkz+(mk^2-(r+\frac{n^2}{4m}))\tau)$ ⇒ summing over all $k\in-\frac{n}{2m}+\mathbb{Z}$, we get

$$\sum_{k\in\mathbb{Z}}\det(r_\alpha t_k)e^{(r_\alpha t_k(\mu),h)}=-q^{-(r+n^2/4m)}\Theta_{-n,m}(\tau,z,u)$$

So: $$\sum_{w\in W}\det(w)e^{(w(\mu),h)}=q^{-r-\frac{n^2}{4m}}\left(\Theta_{n,m}(\tau,z,u)-\Theta_{-n,m}(\tau,z,u)\right) \qquad (\dagger)$$

Applying formula (†) to both the numerator and denominator of RHS of (*), we get:

$$\boxed{ch_{L_\lambda}(h) = q^{-s_\lambda}\cdot\frac{\Theta_{n+1,m+2}(\tau,z,u)-\Theta_{-n-1,m+2}(\tau,z,u)}{\Theta_{1,2}(\tau,z,u)-\Theta_{-1,2}(\tau,z,u)}, \quad \text{where } q:=e^{2\pi i\tau},\ s_\lambda := r+\frac{(n+1)^2}{4(m+2)}-\frac{1}{8}}$$

(follows by recalling that $\rho = 2d + \frac{1}{2}\alpha$).

Let us consider the simplest nontrivial case $\lambda = d$ (i.e. $m=1, n=0, r=0$) $\Rightarrow L_\lambda = L_{w_0}$ — "basic representation". Then $s_\lambda = \frac{1}{12} - \frac{1}{8} = -\frac{1}{24}$ and so:

Cor 1: $$\boxed{ch_{L_d}(h) = q^{1/24}\cdot\frac{\Theta_{1,3}(\tau,z,u)-\Theta_{-1,3}(\tau,z,u)}{\Theta_{1,2}(\tau,z,u)-\Theta_{-1,2}(\tau,z,u)}}$$

To simplify this expression, we will need the following useful product f-la for theta f-s:

Exercise: Prove

$$\boxed{\Theta_{n,m}(\tau,z,u)\cdot\Theta_{n',m'}(\tau,z,u) = \sum_{j\in\mathbb{Z} \bmod (m+m')\mathbb{Z}} d_j^{(m,m',n,n')}(q)\cdot\Theta_{n+n'+2mj,\,m+m'}(\tau,z,u),}$$

where $$\boxed{d_j^{(m,m',n,n')}(q) := \sum_{k\in\frac{m'n-mn'+2jmm'}{2mm'(m+m')}+\mathbb{Z}} q^{mm'(m+m')k^2}}.$$

Lemma 1: $ch_{L_d}(h) = \dfrac{\Theta_{0,1}(\tau,z,u)}{\varphi(q)}$, where $\varphi(q) = \prod_{n>0}(1-q^n)$

► To deduce this f-la from Corollary 1, we need to prove $\boxed{\Theta_{0,1}(\Theta_{1,2}-\Theta_{-1,2}) = q^{1/24}\cdot\varphi(q)\cdot(\Theta_{1,3}-\Theta_{-1,3})}$ (we omit arguments τ, z, u).

By exercise:

$$\Theta_{0,1}\cdot\Theta_{1,2} = \Theta_{1,3}\cdot\sum_{k\in-\frac{1}{12}+\mathbb{Z}} q^{6k^2} + \Theta_{3,3}\cdot\sum_{k\in\frac{1}{4}+\mathbb{Z}} q^{6k^2} + \Theta_{5,3}\cdot\sum_{k\in\frac{7}{12}+\mathbb{Z}} q^{6k^2}$$

$$\Theta_{0,1}\cdot\Theta_{-1,2} = \Theta_{-1,3}\cdot\sum_{k\in\frac{1}{12}+\mathbb{Z}} q^{6k^2} + \Theta_{1,3}\cdot\sum_{k\in\frac{5}{12}+\mathbb{Z}} q^{6k^2} + \Theta_{3,3}\cdot\sum_{k\in\frac{3}{4}+\mathbb{Z}} q^{6k^2}$$

Also: $\Theta_{5,3} = \Theta_{-1,3}$, $\sum_{k\in\lambda+\mathbb{Z}} q^{6k^2} = \sum_{k\in-\lambda+\mathbb{Z}} q^{6k^2}$ $\Rightarrow$

$$\Rightarrow \boxed{\Theta_{0,1}(\Theta_{1,2}-\Theta_{-1,2}) = (\Theta_{1,3}-\Theta_{-1,3})\cdot\left(\sum_{k\in-\frac{1}{12}+\mathbb{Z}} q^{6k^2} - \sum_{k\in\frac{5}{12}+\mathbb{Z}} q^{6k^2}\right)}$$

But: $\sum_{k\in-\frac{1}{12}+\mathbb{Z}} q^{6k^2} - \sum_{k\in\frac{5}{12}+\mathbb{Z}} q^{6k^2} = \sum_{m\in\mathbb{Z}} (-1)^m\cdot q^{(3m^2+m)/2}\cdot q^{1/24}$ (even m correspond to 1st sum, odd m – to 2nd sum)

Remains to use Euler's pentagonal identity (see e.g. [Homework 6, Problem 3(c)] or our discussion next page):

$$\boxed{\varphi(q) = \sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{3m^2+m}{2}} = 1 + \sum_{m\geq 1}(-1)^m\left(q^{\frac{3m^2+m}{2}} + q^{\frac{3m^2-m}{2}}\right)}$$

■

Remark 1: The formula $ch_{L_d}(h) = \dfrac{\Theta_{0,1}(\tau,z,u)}{\varphi(q)}$ is also clear from the explicit realization of this representation constucted in [Homework 9, Problem 4].

Exercise: Work out this computation explicitly!

Let us provide yet another proof of the equality

$$\boxed{\prod_{k\geq 1}(1-u^{k-1}v^{k})(1-u^{k}v^{k-1})(1-u^{k}v^{k}) = \sum_{m\in\mathbb{Z}}(-1)^m\cdot u^{\frac{m(m+1)}{2}}v^{\frac{m(m-1)}{2}}} \qquad (\diamond)$$

which upon substitution $u\mapsto q$, $v\mapsto q^2$ gives another proof of Euler's pentagonal identity used above.
To prove $(\diamond)$, let us look at the Weyl-Kac denominator f-la for $\widehat{\mathfrak{sl}}_2$:

$$\boxed{\prod_{\gamma\in\Delta_+}\left(1-e^{(-\gamma,h)}\right)^{\mathrm{mult}(\gamma)} = \sum_{w\in W}\det(w)\cdot e^{(w(\rho)-\rho,h)}}$$

Recall that $\Delta_+ = \{\alpha+k\delta\}_{k\geq 0}\cup\{k\delta\}_{k\geq 1}\cup\{-\alpha+k\delta\}_{k\geq 1}$, $-\alpha+\delta=\alpha_0$, and $\delta\in\hat{\mathfrak{h}}^*$ corresponds to $K\in\hat{\mathfrak{h}}$ under $\hat{\mathfrak{h}}^*\simeq\hat{\mathfrak{h}}$.
Set $u:=e^{-(K-\alpha,h)}=e^{-(h_0,h)}$, $v:=e^{-(\alpha,h)}=e^{-(h_1,h)}$. Then, we see that:

$$\boxed{\prod_{\gamma\in\Delta_+}\left(1-e^{(-\gamma,h)}\right)^{\mathrm{mult}(\gamma)} = \prod_{k\geq 0}(1-v\cdot u^kv^k)\cdot\prod_{k\geq 1}(1-u^{k+1}v^{k+1})\cdot\prod_{k\geq 0}(1-u\cdot u^kv^k) = \prod_{k\geq 1}(1-u^{k-1}v^k)(1-u^kv^{k-1})(1-u^kv^k)}$$

If $w=t_k \Rightarrow w(\rho)-\rho = 2(d-k\alpha-k^2\cdot K)+\frac{\alpha}{2}+k\cdot K-2d-\frac{\alpha}{2} = -2k\cdot\alpha+(k-2k^2)\cdot K$
$\Rightarrow \det(w)e^{(w(\rho)-\rho,h)} = v^{2k}(uv)^{2k^2-k} = u^{2k^2-k}\cdot v^{2k^2+k} = (-1)^m\cdot u^{\frac{m(m+1)}{2}}v^{\frac{m(m-1)}{2}}$ for $m=-2k$

If $w=r_\alpha t_k \Rightarrow w(\rho)-\rho = 2(d+k\alpha-k^2K)-\frac{\alpha}{2}+k\cdot K-2d-\frac{\alpha}{2} = (2k-1)\alpha+(k-2k^2)\cdot K$
$\Rightarrow \det(w)e^{(w(\rho)-\rho,h)} = -v^{1-2k}\cdot(uv)^{2k^2-k} = -u^{2k^2-k}\cdot v^{2k^2-3k+1} = (-1)^m\cdot u^{\frac{m(m+1)}{2}}v^{\frac{m(m-1)}{2}}$ for $m=2k-1$

$$\Rightarrow \boxed{\sum_{w\in W}\det(w)\cdot e^{(w(\rho)-\rho,h)} = \sum_{m\in\mathbb{Z}}(-1)^m u^{\frac{m(m+1)}{2}}v^{\frac{m(m-1)}{2}}}$$

So: The Weyl-Kac denominator f-la for $\widehat{\mathfrak{sl}}_2$ implies $(\diamond)$.

In what follows, we shall crucially use the following character product formula:

Proposition 1: For $\lambda = md+\frac{n}{2}\alpha$ $(m\geq n\geq 0)$, we have:

$$\boxed{\begin{aligned}
&ch_{L_d}(h)\cdot ch_{L_\lambda}(h) = \sum_{k\in I}\psi_{m,n,k}(q)\cdot ch_{L_{\lambda+d-k\alpha}}(h), \text{ where}\\
&I := \{k\in\mathbb{Z} \mid -\tfrac{1}{2}(m-n+1)\leq k\leq \tfrac{n}{2}\}\\
&\psi_{m,n,k}(q) := \frac{f_k^{(m,n)}(q)-f_{n+1-k}^{(m,n)}(q)}{\varphi(q)}\\
&f_k^{(m,n)}(q) := \sum_{j\in\mathbb{Z}} q^{(m+2)(m+3)j^2+((n+1)+2k(m+2))j+k^2}
\end{aligned}}$$

Exercise: Prove this formula using product f-la of $\Theta_{..}$-functions and Lemma 1.

Now we consider the tensor product $L_d\otimes L_\lambda$ of $\widehat{\mathfrak{sl}}_2$-integrable, unitary irr. reps $\left(\lambda=md+\frac{n}{2}\alpha,\ m\geq n\geq 0\right)$
The resulting tensor product is a level $m+1$ and is unitary. The latter implies

$$\boxed{L_d\otimes L_\lambda = \bigoplus_{\mu\in P_+} L_\mu^{m_\mu}\ ,\quad m_\mu\in\mathbb{Z}_{\geq 0}\text{-almost all zero}}$$

Here each L_μ is an integrable, unitary, level $m+1$ $\widehat{\mathfrak{sl}}_2$-module. [Exercise]
Moreover, the multiplicities m_μ are uniquely determined by expressing $ch_{L_d\otimes L_\lambda}$ via ch_{L_μ}.

According to Proposition 1, we have:

$$\boxed{ch_{L_d \otimes L_\lambda}(h) = ch_{L_d}(h)\cdot ch_{L_\lambda}(h) = \sum_{k\in I}\sum_{j\in\mathbb{Z}} \Delta^j_{m,n,k}\cdot ch_{L_{d+\lambda-k\alpha-jK}}(h)}$$

where $\Delta^j_{m,n,k} \in \mathbb{C}$ are the coefficients of $\psi_{m,n,k}(q)$, i.e. $\psi_{m,n,k}(q) = \sum_{j\in\mathbb{Z}} \Delta^j_{m,n,k}\cdot q^j$.

Remark 2: (a) The representation theoretical meaning of $\Delta^j_{m,n,k}$ as a multiplicity of $L_{d+\lambda-k\alpha-jK}$ in $L_d\otimes L_\lambda$ implies $\Delta^j_{m,n,k}\in\mathbb{Z}_{\geq 0}$ $\forall j$.

(b) Above we used the fact that for $h = 2\pi i\left(\frac{1}{2}z\alpha - \tau d + uK\right)$ and $j\in\mathbb{Z}$:

$$\boxed{ch_{L_{\mu-jK}}(h) = ch_{L_\mu}(h)\cdot \exp(2\pi i\tau\cdot j) = q^j\cdot ch_{L_\mu}(h)}$$

(c) In part (b), we actually used that $L_{d+\lambda-k\alpha-jK}$ remains irreducible as an $\hat{sl}_2$-module.

Cor 2: $\boxed{L_d\otimes L_\lambda \simeq \bigoplus_{\substack{k\in I\\ j\in\mathbb{Z}}} L_{d+\lambda-k\alpha-jK}^{\oplus\Delta^j_{m,n,k}}}$ as $\hat{sl}_2$-modules

For what follows, we need to determine the minimal value of j such that $\Delta^j_{m,n,k}>0$ (for given m,n,k).

Lemma 2: Define $\tau, s\in\mathbb{Z}$ via $\tau := \begin{cases} n+1, & \text{if } k\geq 0\\ m-n+1, & \text{if } k<0\end{cases}$, $s := \begin{cases} n+1-2k, & \text{if } k\geq 0\\ m-n+2+2k, & \text{if } k<0\end{cases}$, so that $1\leq s\leq \tau\leq m+1$ for $m\geq n\geq 0$ and $k\in I$.

Then:

$$\boxed{\begin{aligned}
&\varphi(q)\cdot q^{-k^2}\cdot\psi_{m,n,k}(q) = A+B+C, \text{ where}\\
&A = 1-q^{\tau s} - q^{(m+2-\tau)(m+3-s)}\\
&B = \sum_{j>0} q^{(m+2)(m+3)j^2 + ((m+3)\tau-(m+2)s)j}\cdot\left(1-q^{2(m+2)sj+\tau s}\right)\\
&C = \sum_{j>0} q^{(m+2)(m+3)j^2 - ((m+3)\tau-(m+2)s)j}\cdot\left(1-q^{2(m+2)(m+3-s)j+(m+2-\tau)(m+3-s)}\right)
\end{aligned}}$$

Exercise: Prove this! (straightforward computation)

Lemma 3: Given m,n,k as above $\min\{j \mid \Delta^j_{m,n,k}\neq 0\} = k^2$ and $\Delta^{k^2}_{m,n,k} = 1$.

► By Lemma 2:

$$\psi_{m,n,k}(q) = q^{k^2}\cdot\frac{1}{\varphi(q)}\cdot(A+B+C).$$

Clearly $A+B+C = 1-q^{\tau s} - q^{(m+2-\tau)(m+3-s)}$ + higher powers of q

$1/\varphi(q) = 1+\sum_{j>0} p(j)q^j$

$\Rightarrow$ $\psi_{m,n,k}(q) = q^{k^2}$ + higher powers of q.

Exercise: Explain why all $\Delta^j_{m,n,k}\in\mathbb{Z}_{\geq 0}$ without referring to representation theoretical meaning of $\Delta^j_{m,n,k}$.

Now we are ready to establish unitarity of Virasoro discrete series!

Let $U^{(j)}_{m,n,k} := \{\widehat{sl}_2\text{-highest weight vectors in } L_d \otimes L_\lambda \text{ of highest weight } d+\lambda-k\alpha-j\cdot K\}$

$U_{m,n,k} := \bigoplus_{j\in\mathbb{Z}} U^{(j)}_{m,n,k} = \{sl_2\text{-highest weight vectors in } L_d\otimes L_\lambda \text{ of highest weight } d+\lambda-k\alpha\}$

By above: $\dim U^{(j)}_{m,n,k} = \Delta^{j}_{m,n,k}$, $tr_{U_{m,n,k}}(q^{d}) = \psi_{m,n,k}(q)$.

Recalling the coset construction of Goddard-Kent-Olive (Lecture 13), we have a Virasoro action on $L_d\otimes L_\lambda$, which commutes with $\widehat{sl}_2$-action and thus acts on $U_{m,n,k}$ (for each $k\in I$)

$$\text{Vir} \curvearrowright U_{m,n,k}$$

The central charge was computed in the end of Lecture 13 and we got: $c = 1-\frac{6}{(m+2)(m+3)}$

Let us now compute the action of L_0 on $U_{m,n,k}$.

According to [Homework 10, Problem 2], we have $\Delta = 2(k+h^\vee)(L_0+d)$ (Δ = Casimir, k = level, L_0 = Sugawara operator) and $\Delta|_{L(\mu)} = (\mu,\mu+2\rho)\cdot \mathrm{Id}_{L(\mu)}$.

Applying Goddard-Kent-Olive construction to $\widehat{sl}_2$-modules $V'=L_d$, $V''=L_\lambda$, we see that $L_0 \curvearrowright V'\otimes V''$ via (note: $h^\vee=2$)

$$L_0 = \left(\frac{(d,d+2\rho)}{2(1+2)}-d\right)\otimes 1 + 1\otimes\left(\frac{(\lambda,\lambda+2\rho)}{2(m+2)}-d\right)\otimes 1 - \left(\frac{\Delta|_{L_d\otimes L_\lambda}}{2(m+3)} - d\otimes 1 - 1\otimes d\right) \underset{\rho=2d+\frac{1}{2}\alpha}{\overset{\lambda=md+\frac{n}{2}\alpha}{=}} \frac{n(n+2)}{4(m+2)} - \frac{\Delta}{2(m+3)},$$ where Δ = Casimir operator $\curvearrowright L_d\otimes L_\lambda$

Recalling the above formula $\Delta = 2(k+h^\vee)(L_0+d)$ on each irreducible summand L_μ as well as $k=m+1$, $h^\vee=2$, get:

$$\Delta = 2(m+3)d + \Delta_0 + 2\sum_{\substack{a\in B'\\ m>0}} a_{-m}a_m\,,\quad \Delta_0 = ef+fe+\frac{\alpha^2}{2} = \frac{\alpha^2}{2}+\alpha+2fe = \text{Casimir of } sl_2$$

But any $v\in U_{m,n,k}$ is a h.wt. $\widehat{sl}_2$-vector $\Rightarrow a_m(v)=0$ $\forall a\in sl_2, m>0$. Also: $e(v)=0$.

So: $\Delta(v) = \left(2(m+3)d + \frac{\alpha^2}{2}+\alpha\right)(v)$ for $v\in U_{m,n,k}$.

As $wt(v) = d+\lambda-k\alpha = (m+1)d + \frac{n-2k}{2}\alpha \Rightarrow \left(\frac{\alpha^2}{2}+\alpha\right)(v) = \left(\frac{(n-2k)^2}{2}+(n-2k)\right)\cdot v$

$\Rightarrow$ $\Delta(v) = \left(2(m+3)d + \frac{(n-2k)^2}{2}+(n-2k)\right)v$, $v\in U_{m,n,k}$

Thus:
$$L_0 = \frac{n(n+2)}{4(m+2)} - d - \frac{(n-2k)(n-2k+2)}{4(m+3)} \quad \text{on } U_{m,n,k}$$

But combining Lemma 3 with $tr_{U_{m,n,k}}(q^{-d}) = \psi_{m,n,k}(q)$, we see that the minimal eigenvalue of $-d|_{U_{m,n,k}}$ is k^2.

<u>Cor 3</u>: The minimal eigenvalue of $L_0|_{U_{m,n,k}}$ is

$$h = k^2 + \frac{n(n+2)}{4(m+2)} - \frac{(n-2k)(n-2k+2)}{4(m+3)} = \frac{((m+3)r-(m+2)s)^2-1}{4(m+2)(m+3)}$$

Recall that in Sugawara construction if M is a unitary $\hat{\mathfrak{g}}$-module $\Rightarrow M$ is a unitary Vir-module (Lecture 13 Prop. 2)

$\Rightarrow$ in coset construction if V', V'' - unitary $\hat{\mathfrak{a}}$-modules $\Rightarrow V'\otimes V''$ is a unitary Vir-module.

Thus, $U_{m,n,k}$ - unitary Vir-module $\Rightarrow$ Vir-submodule of $U_{m,n,k}$ generated by L_0-eigenvector with eigenvalue h is an <u>irreducible unitary highest weight representation of Vir</u> with

$$c = c(m) = 1-\frac{6}{(m+2)(m+3)}\,,\quad h = h_{r,s}(m) = \frac{((m+3)r-(m+2)s)^2-1}{4(m+2)(m+3)}$$

As we vary $0\le n\le m$ and $k\in I$, we get all possible $1\le s\le r\le m+1$.

<u>Theorem 1</u>: For any $1\le s\le r\le m+1$, Virasoro irreducible module $L_{c(m),h_{r,s}(m)}$ is unitary!

Remark 3: (a) For a fixed $m\in\mathbb{Z}_{>0}$, $\left(L_{\frac{1}{2}}\otimes\bigoplus_{\substack{n\in\mathbb{Z}_{\geq 0}\\ 0\leq n\leq m}} L_{md+\frac{n}{2}d}\right)^{\widehat{\mathfrak{sl}_2}_+} \underset{Vir-mod}{\simeq} \bigoplus_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\leq s\leq r\leq m+1}} U^{(m)}_{r,s}$, where the highest component of $U^{(m)}_{r,s}$ is $L_{c(m),h_{r,s}(m)}$ - representation of the discrete series

(b) $ch_{L_{c(m),h_{r,s}(m)}} \leq \frac{q^{h_{r,s}(m)}}{\varphi(q)}\cdot\left(1-q^{rs}-q^{(m+2-r)(m+3-s)}+B+C\right)$, where B,C are as in Lemma 2.

FACT (Feigin-Fuchs 1984): $U^{(m)}_{r,s}$ is irreducible Vir-module, i.e. $U^{(m)}_{r,s}\simeq L_{c(m),h_{r,s}(m)}$.

We are not going to prove this rather hard result.

Let us conclude today's lecture by proving [Lecture 11, Theorem 2]:

Theorem 2: For $r,s\geq 1$, $\det_{rs}(c,h_{r,s}(c))=0$, where $h_{r,s}(c)=\frac{1}{48}\left((13-c)(r^2+s^2)+\sqrt{(c-1)(c-25)}(r^2-s^2)-24rs-2+2c\right)$

↑ determinant of the Shapovalov form on degree $-rs$ component of Virasoro Verma module.

Recall: This result was crucially used in our proof of the determinant formula

$$\det_m(c,h)=K_m\cdot\prod_{\substack{r,s\geq 1\\ rs\leq m}}(h-h_{r,s}(c))^{p(m-rs)}$$

← [Lecture 11, Theorem 3].

We have
$$\left.\begin{aligned} ch_{L_{c(m),h_{r,s}(m)}} &\leq \frac{q^{h_{r,s}(m)}}{\varphi(q)}\left(1-q^{rs}-q^{(m+2-r)(m+3-s)}+\text{higher powers of } q\right)\\ ch_{M_{c(m),h_{r,s}(m)}} &= \frac{q^{h_{r,s}(m)}}{\varphi(q)}\end{aligned}\right\}\Rightarrow$$

$\Rightarrow J_{c(m),h_{r,s}(m)} := \mathrm{Ker}(M_{c(m),h_{r,s}(m)} \twoheadrightarrow L_{c(m),h_{r,s}(m)})$ has a nonzero component at each level n such that $n\geq\min\{r\cdot s, \underbrace{(m+2-r)}_{=:r'}\underbrace{(m+3-s)}_{=:s'}\}$

So: $\det_n(c_m,h)$ as a polynomial in h has a zero at each $h=h^{(m)}_{r,s}$ with $1\leq r,s\leq m+1$ and $rs\leq n$.

Define
$$\varphi_{r,s}(c,h):=\begin{cases}(h-h_{r,s}(c))(h-h_{s,r}(c)), & \text{if } r\neq s\\ h-h_{r,r}, & \text{if } r=s.\end{cases}$$

Then clearly $\varphi_{r,s}(c,h)\in\mathbb{C}[c,h]$ is irreducible.

By above, $\det_n(c,h)$ vanishes at infinitely many points $\{(c(m),h_{r,s}(m))\,|\,m\geq\max\{r,s\}-1\}$ as long as $n\geq rs$.

$\Rightarrow \det_n(c,h)$ is divisible by $\varphi_{r,s}(c,h)$ for any r,s s.t. $n\geq rs$. $\Rightarrow \det_{rs}(c,h_{r,s}(c))=0 \Rightarrow$ Theorem 2.

✓